Some Topological Properties of Discrete Surfaces

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Abstract. A basic property of a simple closed surface is the Jordan's property: the complement of the surface has two connected components. We call *back-component* any such component, and the union of a back-component and the surface is called the *closure* of this back-component. We introduce the notion of *strong surface* as a surface which satisfies a strong homotopy property: the closure of a back-component is strongly homotopic to that back-component. This means that we can homotopically remove any subset of a strong surface from the closure of a back-component. On the basis of some results on homotopy ([2]), and strong homotopy ([3], [4], [5]), we have proved that the simple closed 26-surfaces defined by Morgenthaler and Rosenfeld ([19]), and the simple closed 18-surfaces defined by Malgouyres ([15]) are both strong surfaces. Thus, strong surfaces appear as an interesting generalization of these two notions of a surface.

1 Introduction

Intuitively, a simple closed surface may be seen as a closed surface which is not *singular*, i.e., as a closed surface which "does not fold upon itself" (see Fig. 1). In the continuous spaces, it is possible to detect locally the singularities of a surface, this leads to a local characterization of simple closed surfaces. In Z^3, several approaches of simple closed surfaces have been proposed:
- a graph-theoretical approach: a surface is defined as a thin set of points linked by adjacency relations ([10], [14], [15], [19]);
- a voxel approach: a surface is defined as a set of faces (surfels) between pairs of adjacent voxels ([1], [8]);
- a general topology approach ([12]);
- a combinatorial approach: a surface is defined as a structure ([7], [13]).
In this paper, we consider surface definitions based on the graph-theoretical approach. These surfaces are connected sets which share two basic properties:

$\mathcal{P}_1$: They satisfy the Jordan property, i.e, the complement of a surface X is made of two connected components called *back-components*, we will denote A and B these components;

$\mathcal{P}_2$: The neighborhood of each point x of X contains two connected components which are included in the complement of X and which are adjacent to x.

Let us call $\mathcal{S}$ the class of surfaces which satisfy $\mathcal{P}_1$ and $\mathcal{P}_2$. It was proved ([17], [18]) that $\mathcal{S}$ cannot be locally characterized. Moreover, it was proved that $\mathcal{S}$ contains some singular surfaces such as a "pathological pinched torus". Let us consider the pinched torus of Fig. 1 (b). In the continuous space, such a surface does not satisfy the equivalent of Property $\mathcal{P}_2$: an arbitrary small neighborhood of the singular point x of the pinched torus contains three connected components of the complement which are adjacent to x. Nevertheless, it was shown that there exists a pinched torus in Z^3 such that $\mathcal{P}_1$ and $\mathcal{P}_2$ are satisfied (Fig. 2, set X_1).

Let us examine if it is possible to define a property $\mathcal{P}_3$ such that the class of sets which belong to $\mathcal{S}$ and which satisfy Property $\mathcal{P}_3$ does not contain singular surfaces. More precisely, let us investigate some property based upon homotopy. Again, we consider the singular pinched torus of Fig. 1 (b), we denote by A its finite back-component: we note that $A \cup X$ has a hole which does not exist in A, in other words, $A \cup X$ and A are not homotopic. Let us call the *closure* of a back-component the union of the surface and that back-component. We consider the class $\mathcal{H}$ composed of the surfaces of $\mathcal{S}$ such that any back-component of these surfaces is homotopic to its closure. We could think that $\mathcal{H}$ does not contain singular surfaces. In fact, we proved that this is not the case. Let us consider the pinched sphere X of Fig. 1 (c). We note that $A \cup X$ is homotopic to A. In the continuous space, the pinched sphere does not satisfy the equivalent of Property $\mathcal{P}_2$: an arbitrary small neighborhood of the singular points of the pinched sphere contains three or four connected components of the complement. Nevertheless, we showed that there exists a pinched sphere in Z^3 which satisfies $\mathcal{P}_1$ and $\mathcal{P}_2$ and which belongs to $\mathcal{H}$ (Fig. 2, set X_3). It follows that $\mathcal{H}$ cannot be locally characterized.

If we look further at the singular pinched sphere, we note that $A \cup X$ is homotopic to A, but the removal of a subset of X can break the homotopy of $A \cup X$ (the removal of a part of the central segment generates a hole). It follows the idea of considering the property of strong homotopy ([3], [4]) for eliminating singular surfaces which belong to $\mathcal{S}$. We define the class of *strong surfaces* which is composed of all surfaces X of $\mathcal{S}$ such that we can homotopically remove any subset of the surface from the closure of a back-component. The main results of this paper show that two notions of surfaces proposed in the litterature, the Morgenthaler's 26-surface ([19]) and the Malgouyres' surfaces ([15]), satisfy this strong homotopy property.

The two interesting points concerning these results are the following:

1) We have a new global property for existing surfaces. In fact, perhaps the only global property of discrete surfaces which appears in the litterature is the Jordan's property (Property $\mathcal{P}_1$). The proposed new property allows to make a link between surfaces and the global notion of homotopy.

2) It has been pointed out ([14]) that there is a need for a new definition of surfaces which can better detect thin sets which appear in real images. In this context, strong surfaces can bring a new insight into the notion of surface and they may be a step towards a more general definition.

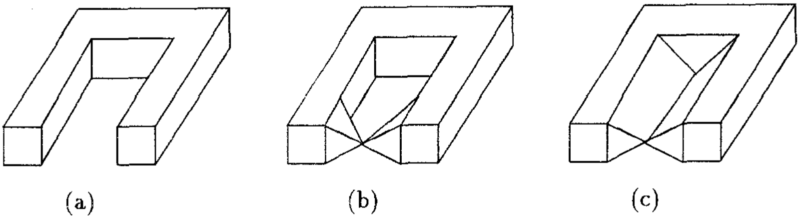

Fig. 1. A simple closed surface (a), and two singular closed surfaces: a pinched torus (b), and a pinched sphere (c).

2 Basic notions

We recall some basic notions of 3D discrete topology (see also [11]).
We denote $E = Z^3$, Z being the set of relative integers. A point $x \in E$ is defined by (x_1, x_2, x_3) with $x_i \in Z$. We consider the four neighborhoods:
$N_{124}(x) = \{x' \in E;\ Max[|x_1 - x'_1|, |x_2 - x'_2|, |x_3 - x'_3|] \leq 2\}$,
$N_{26}(x) = \{x' \in E;\ Max[|x_1 - x'_1|, |x_2 - x'_2|, |x_3 - x'_3|] \leq 1\}$,
$N_{18}(x) = \{x' \in E;\ |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leq 2\} \cap N_{26}(x)$.
$N_6(x) = \{x' \in E;\ |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leq 1\}$,
We denote $N^*_k(x) = N_k(x) \setminus \{x\}$, with $k = 6$, 18, 26, 124.
Two points x and y are said to be *n-adjacent* (n = 6, 18, 26) if $y \in N^*_n(x)$.
An *n-path* π is a sequence of points $x_0..x_k$, with x_i n-adjacent to x_{i-1}, for $i = 1..k$. The *length of* π is equal to k. The path is *elementary* if all points of the sequence are different except possibly $x_0 = x_k$. An elementary n-path π is *simple* if each point of π has, at most, two n-adjacent points in π. If $x_0 = x_k$, π is *closed*. Note that any path from x_0 to x_k contains an elementary path and a simple path from x_0 to x_k.
A set of points is an *n-curve* if the points of the set can be arranged in such a way that they form an n-path. Elementary, simple, closed n-curves may be defined in a similar way.

An object $X \subset E$ is said to be *n-connected* if for any two points of X, there is an n-path in X between these two points. The equivalence classes relative to this relation are the *n-connected components of* X. The set composed of all the n-connected components of X is denoted $\mathcal{C}_n(X)$. The set of all n-connected components of X n-adjacent to a point x is denoted $\mathcal{C}^x_n(X)$. Note that $\mathcal{C}_n(X)$ and $\mathcal{C}^x_n(X)$ are sets of subsets of X and not sets of points.
As in 2D, if we use an n-connectivity for X we have to use another $\overline{n}$-connectivity for $\overline{X}$, i.e. the 6-connectivity for X is associated to the 18 or the 26 connectivity for $\overline{X}$ (and vice versa). This is necessary for having a correspondence between the topology of X and the topology of $\overline{X}$. Furthermore, it is sometimes necessary to distinguish the 6-connectivity associated with the 18-connectivity and the 6-connectivity associated with the 26-connectivity. Whenever we will have to make this distinction, a 6^+-notion will indicate a 6-notion associated with the

18-connectivity. So, we can have $(n, \overline{n}) = (6, 26)$, $(26, 6)$, $(6^+, 18)$ or $(18, 6^+)$. Note that, if X is finite, the infinite $\overline{n}$-connected component of $\overline{X}$ is the *background*, the other $\overline{n}$-connected components of $\overline{X}$ are the *cavities*.

Let γ and γ' be two closed n-paths in X. We say that γ' is *an elementary deformation* of γ, which is noted $\gamma \sim \gamma'$, if γ and γ' are the same but in a little portion P:
- for $n = 6$, P is an unit square (a 2×2 square);
- for $n = 6^+$, 18, 26, P is an unit cube (a $2 \times 2 \times 2$ cube).
We say that γ' is a *deformation* of γ if there is a sequence of closed n-paths $\gamma_0..\gamma_k$ such that $\gamma = \gamma_0$, $\gamma' = \gamma_k$ and $\gamma_{i-1} \sim \gamma_i$ for $i = 1..k$.
The presence of an n-hole in X is detected whenever there is a closed n-path in X that cannot be deformed in X to a single point. For example a hollow torus has one cavity and two holes.

3 Homotopy and strong homotopy

In this section, we recall some notions of homotopy and strong homotopy. The homotopy in a discrete grid may be defined through the notion of simple point. A point $x \in X$ is said to be *n-simple (for X)* if its removal does not change the topology of the image, i.e., if there is a one to one correspondence between the n-connected components of X, the $\overline{n}$-connected components of $\overline{X}$, the n-holes of X, and the n-connected components of $X \setminus \{x\}$, the $\overline{n}$-connected components of $\overline{X} \cup \{x\}$, the n-holes of $X \setminus \{x\}$, respectively.
The set Y is *(lower) n-homotopic* to the set X if Y may be obtained from X by deleting n-simple points. If Y is lower n-homotopic to X, the set $S = X \setminus Y$ is called a *(lower) n-simple set.*

Let $X \subset E$ and $x \in E$. The *geodesic n-neighborhood of x inside X of order k* is the set $N_n^k(x, X)$ defined recursively by:
$N_n^1(x, X) = N_n^*(x) \cap X$ and $N_n^k(x, X) = \cup\{N_n(y) \cap N_{26}^*(x) \cap X, y \in N_n^{k-1}(x, X)\}$.
In other words $N_n^k(x, X)$ is the set composed of all points y of $N_{26}^*(x) \cap X$ such that there exists an n-path π from x to y of length less than or equal to k, all points of π, except possibly x, belonging to $N_{26}^*(x) \cap X$. We consider the following neighborhoods; let $X \subset E$ and $x \in E$:
The *geodesic neighborhoods* $G_n(x, X)$ are defined by:

$$G_6(x, X) = N_6^2(x, X); \quad G_{6^+}(x, X) = N_6^3(x, X);$$
$$G_{18}(x, X) = N_{18}^2(x, X); \quad G_{26}(x, X) = N_{26}^1(x, X).$$

We give now a definition of topological numbers ([2]).
Let X be a subset of E and x be a point of E.
The *topological numbers* $T_n(x, X)$ are defined by:

$$T_6(x, X) = \#\mathcal{C}_6[G_6(x, X)]; \quad T_{6^+}(x, X) = \#\mathcal{C}_6[G_{6^+}(x, X)];$$
$$T_{18}(x, X) = \#\mathcal{C}_{18}[G_{18}(x, X)]; \quad T_{26}(x, X) = \#\mathcal{C}_{26}[G_{26}(x, X)].$$

These numbers lead to a characterization of simple points ([2]).
Let X be a subset of E and x be a point of X:
x is an n-simple point $\Leftrightarrow$ $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$.

We introduce the notion of strong homotopy: see [3], [4], see also [9] where the notion of hereditarily simple set is introduced, this notion is equivalent to the notion of strongly simple set presented hereafter:

Definition 1. Let $X \subset E$ and $Y \subset X$. The set Y is *strongly (lower) n-homotopic to* X if, for each subset Z such that $Y \subset Z \subset X$, Z is lower n-homotopic to X. If Y is strongly n-homotopic to X, we say that $X \setminus Y$ is a *strongly (lower) n-simple set.*

The notion of P-simple point allows to handle strong homotopy ([3], [4]):

Definition 2. Let $X \subset E$.
Let $P \subset X$ and $x \in P$: x is *P_n-simple (for X)* if, for each subset S of $P \setminus \{x\}$, x is n-simple for $X \setminus S$. We denote $S_n(P)$ the set of all P_n-simple points.
A set D is *P_n-simple (for X)* if D is a subset of $S_n(P)$.

Let $P \subset X$ and $D \subset S_n(P)$. We denote $D = \{x_1, .., x_k\}$. Suppose we delete in sequence the points $x_1, .., x_k$. At the step i of this procedure, since x_i is P-simple, x_i is simple for the set $X \setminus D_i$ with $D_i = \{x_1, .., x_{i-1}\}$ and we do not change the homotopy by deleting x_i. Hence X and $X \setminus D$ are homotopic. So, we have; let $X \subset E$ and $P \subset X$:

for each subset D of $S_n(P)$, $X \setminus D$ is n-homotopic to X.

In other words:

the set $X \setminus S_n(P)$ is strongly n-homotopic to X.

Let $Y \subset X$ be a set strongly n-homotopic to X. Let $P = X \setminus Y$. Let $x \in P$. For each subset S of $P \setminus \{x\}$, the set $X \setminus S$ is n-homotopic to X and the set $X \setminus (S \cup \{x\})$ is n-homotopic to X (From Def. 1). Hence, x is n-simple for $X \setminus S$ (from definitions of simple points and homotopic sets). So, x is P_n-simple and P is a P_n-simple set. It follows the property:

Theorem 3. *Let X be subset of E. A set $Y \subset X$ is strongly n-homotopic to X if and only if the set $P = X \setminus Y$ is a P_n-simple set for X.*

Since the characterization of a simple point is local, we can derive a local characterization of P-simple points; since $G_n(x, X) \subset N^*_{26}(x)$ and $G_{\overline{n}}(x, \overline{X}) \subset N^*_{26}(x)$, we have:

$$x \text{ is } P_n\text{-simple} \Leftrightarrow \forall S \subset P \cap N^*_{26}(x),\ x \text{ is } n\text{-simple for } X \setminus S.$$

Thus, the checking for P-simple points could be made by checking the condition $[T_n(x, X \setminus S) = 1$ and $T_{\overline{n}}(x, \overline{X \setminus S}) = 1]$ for all sets $S \subset P \cap N^*_{26}(x)$. The following property allows to have a more efficient characterization ([3], [4], [5]):

Theorem 4. *Let $P \subset X$ and $x \in P$; we denote $R = X \setminus P$:*

$$x \text{ is } P_n\text{-simple for } X \Leftrightarrow \begin{cases} 1)\ T_n(x, R) = 1 \\ 2)\ T_{\overline{n}}(x, \overline{X}) = 1 \\ 3)\ \forall y \in N^*_n(x) \cap P,\ T_n(x, R \cup \{y\}) = 1 \\ 4)\ \forall y \in N^*_{\overline{n}}(x) \cap P,\ T_{\overline{n}}(x, \overline{X} \cup \{y\}) = 1 \end{cases}$$

4 Strong surfaces

Definition 5. Let $X \subset E$ be an n-connected set. We denote $|X|^x = N_{26}^*(x) \cap X$. The set X is an *n-thin set* if, $\forall x \in X$, $\#\mathcal{C}_{\overline{n}}^x(|\overline{X}|^x) = 2$. Let X be an n-thin set. The set X is an *n-separating set* if $\overline{X}$ has two $\overline{n}$-connected components.
If X is an n-separating set, we will denote A and B the two components of $\overline{X}$. These components are called the *back-components* of X. The *closure* of a back-component is the union of that back-component and X. Furthermore, if $\forall x \in X$, x is $\overline{n}$-adjacent to A and B, we say that X is a *strongly n-separating set.* We denote $\mathcal{S}_n = \{X \subset E$, such that X is an n-separating set $\}$.

As mentioned in the introduction, all proposed notions of simple closed n-surfaces correspond to sets which belong to $\mathcal{S}_n$. It is therefore interesting to see if there is a local characterization of $\mathcal{S}_n$.
A *configuration* is any subset of a $5 \times 5 \times 5$ cube. Let $\mathcal{E} = \{X \subset E\}$. We say that a class $\mathcal{F} \subset \mathcal{E}$ admits a *local characterization* if there is a set $\mathcal{K}$ of configurations, such that $X \in \mathcal{F} \Leftrightarrow \forall x \in X$, $(N_{124}(x) \cap X) \in \mathcal{K}$. It was shown that ([17], [18]):

The class $\mathcal{S}_{26}$ does not admit a local characterization

Let us consider the Fig. 2 where three 26-thin sets are depicted: X_1 is a "pinched torus", X_2 is a "double sphere", X_3 is a "pinched sphere". The points which belong to these sets are labelled by the capital letters A,...,X. If two points are labeled by the same letter and if this letter is different from the letter "X", then their corresponding 124-neighborhoods are the same up to an isometry.
It may be seen that $X_1 \in \mathcal{S}_{26}$ and $X_2 \notin \mathcal{S}_{26}$. Nevertheless all the configurations of X_2 are appearing in X_1. This illustrates the above mentioned property: if the class $\mathcal{S}_{26}$ was admitting a local characterization $\mathcal{K}$, all the configurations appearing in X_1 would belong to $\mathcal{K}$, and then X_2 would belong to $\mathcal{S}_{26}$.
Note that this same example also shows that the class of strongly 26-separating sets cannot be locally characterized.
Hence, we may wonder if there is an additional global property which is natural for surfaces such that the class composed of the sets of $\mathcal{S}_n$ satisfying this property could be locally characterized. Intuitively, we would like to eliminate, from $\mathcal{S}_n$, the surfaces which "locally fold upon themselves", i.e., which are "singular". The pinched torus X_1 is an example of a singular surface. The presence of such sets in $\mathcal{S}_n$ is the reason why this class cannot be locally characterized. Let us examine if we can use the notion of homotopy for eliminating singular surfaces from $\mathcal{S}_n$. We first consider the class $\mathcal{H}_n$ composed of all sets X belonging to $\mathcal{S}_n$, such that any back-component of X is n-homotopic to its closure.
Again, let us see the Fig. 2. We denote A_1 the finite back-component of $\overline{X}_1$ (i.e, the cavity of X_1). We note that the set X_1 does not belong to $\mathcal{H}_{26}$: $X_1 \cup A_1$ has one hole while A_1 does not. Nevertheless, $\mathcal{H}_n$ contains some singular surfaces. This point may be seen through the following property:

The class $\mathcal{H}_{26}$ does not admit a local characterization

This property may be proved by considering the set X_3 of Fig. 2. We denote A_3 the cavity of X_3. It may be seen that $X_3 \in \mathcal{H}_{26}$. Nevertheless, all the configurations of the points of X_2 are appearing in X_3.

	1 st plane	2 nd plane	3 nd plane	4 th plane
X_1	• • • • • • • • • G H • • H G • • I J • • J I • • X X • • X X • • X X • • X X • • X • • • • X • • X • • • • X • • X X • • X X • • X X X X X X • • • • • • • • •	• A B • • B A • C • • E E • • C D • • F F • • D X • • X X • • X X • • X X • • X X • X • • X • X X • X • • X • X X • • X X • • X X • • • • • • X • X X X X X X •	• A B • • B A • C • • E E • • C D • • F F • • D X • • X X • • X X • • X X • • X X • X • • X • X X • X • • X • X X • • X X • • X X • • • • • • X • X X X X X X •	• • • • • • • • • G H • • H G • • I J • • J I • • X X • • X X • • X X • • X X • • X • • • • X • • X • • • • X • • X X • • X X • • X X X X X X • • • • • • • • •
X_2	• • • • • • • • • G H • • H G • • I J • • J I • • I J • • J I • • G H • • H G • • • • • • • • •	• A B • • B A • C • • E E • • C D • • F F • • D D • • F F • • D C • • E E • • C • A B • • B A •	• A B • • B A • C • • E E • • C D • • F F • • D D • • F F • • D C • • E E • • C • A B • • B A •	• • • • • • • • • G H • • H G • • I J • • J I • • I J • • J I • • G H • • H G • • • • • • • • •
X_3	• • • • • • • • • G H • • H G • • I J • • J I • • X X • • X X • • X X • • X X • • X X • • X X • • X X X X X X • • • • • • • • •	• A B • • B A • C • • E E • • C D • • F F • • D X • • X X • • X X • • X X • • X X • • X X • • X X • • • • • • X • X X X X X X •	• A B • • B A • C • • E E • • C D • • F F • • D X • • X X • • X X • • X X • • X X • • X X • • X X • • • • • • X • X X X X X X •	• • • • • • • • • G H • • H G • • I J • • J I • • X X • • X X • • X X • • X X • • X X • • X X • • X X X X X X • • • • • • • • •

Fig. 2. Examples of 26-thin sets.

The problem is that $X_3 \in \mathcal{H}_{26}$, but X_3 is a singular set. Let us consider the set C composed of the four underlined points in X_3. We note that $(A_3 \cup X_3) \setminus C$ is not 26-homotopic to A_3 (a hole is created). Hence, $A_3 \cup X_3$ is 26-homotopic to A_3, but the deletion of a subset of X_3 can change the homotopy of $A_3 \cup X_3$. It follows the idea to use the notion of strong homotopy for characterizing surfaces.

Definition 6. Let $X \subset E$ be an n-separating set. The set X is a *strong (closed) n-surface* if any back-component of X is strongly n-homotopic to its closure.

From Th. 3 and Th. 4, it is straighforward to derive a characterization which enables to detect if an n-separating set X is a strong n-surface:

Theorem 7. *Let $X \subset E$ be an n-separating set.*
The set X is a strong n-surface if and only if for each x of X:
1) $T_n(x, |A|^x) = 1$ and $T_n(x, |B|^x) = 1$;
2) $T_{\overline{n}}(x, |A|^x) = 1$ and $T_{\overline{n}}(x, |B|^x) = 1$;
3) $\forall y \in N_n^*(x) \cap X$, $T_n(x, |A|^x \cup \{y\}) = 1$ and $T_n(x, |B|^x \cup \{y\}) = 1$;
4) $\forall y \in N_{\overline{n}}^*(x) \cap X$, $T_{\overline{n}}(x, |A|^x \cup \{y\}) = 1$ and $T_{\overline{n}}(x, |B|^x \cup \{y\}) = 1$.

We see that this characterization is not "fully local": the knowledge of $|X|^x$ is not sufficient to decide if x satisfies the four above properties, we also need to know the distribution of the points of $|\overline{X}|^x$ between $|A|^x$ and $|B|^x$. In fact, since the symmetry of the four conditions with respect to A and B, we see that it is sufficient to know this distribution up to a renaming of A and B. More precisely, it is sufficient to have the knowledge of an assignment. Let $X \subset E$ be an n-separating set. The datum of an *assignment* on X is the datum, for each x belonging to X, of a map f_x: $|\overline{X}|^x \longrightarrow \{0,1\}$ such that $\{f_x^{-1}(0), f_x^{-1}(1)\} = \{|A|^x, |B|^x\}$.

Let us consider the neighborhoods depicted in Fig. 3 (a)...(h). We denote by x the central point of these neighborhoods. We suppose that each of these configurations is a subset of a strong n-separating set. We will examine if it is possible that these configurations belong to a strong n-surface, with $n = 18$ or 26, i.e., if x could satisfy the four properties of Th. 7. Let us first note that, for all these configurations, we have $\#\mathcal{C}^x_{\overline{n}}(|\overline{X}|^x) = 2$. We denote $\mathcal{C}^x_{\overline{n}}(|\overline{X}|^x) = \{A^{xx}, B^{xx}\}$, with $A^{xx} \subset A$ and $B^{xx} \subset B$.

We first consider the configurations (a),...,(e). We note that they do not contain any component of $\overline{X}$ not $\overline{n}$-adjacent to x, i.e., we have $A^{xx} = |A|^x$ and $B^{xx} = |B|^x$. Then, it is possible to check directly if these configurations satisfy the 4 conditions of Th. 7 (the data of an assignment is not necessary). We see that the configurations (a), (b), (c), satisfy these conditions for $n = 18$ or $n = 26$. The configuration (d) satisfies the conditions for $n = 18$ but not for $n = 26$ (the condition 4 is not satisfied). The configuration (e) satisfies the conditions for $n = 18$ but not for $n = 26$ (the condition 3 is not satisfied).

We now consider the configurations (f),...,(h). These configurations contain components of $\overline{X}$ not $\overline{n}$-adjacent to x. It may be seen that for $n = 18$, configuration (f) could belong to a strong surface only if the two points u and v belong to the same component of $\overline{X}$ (i.e., A or B); otherwise the condition 3 would not be satisfied. In this case we will say that we have a *strong assignment*. For $n = 26$, the configuration (f) satisfies the four conditions for strong surfaces if $v \in A$ or if $v \in B$. By examining the other configurations, we have the following result:

- the central points of the configurations (a), (b), (c), (f), satisfy the conditions for strong 26-surfaces. The other ones cannot satisfy these conditions;
- the central points of the configurations (a), (b), (c), (d), (e), (h), satisfy the conditions for strong 18-surfaces. The other ones satisfy the conditions for strong 18-surfaces provided that we have strong assignments.

5 Strong surfaces and Morgenthaler's 26-surfaces

We present the definition of surfaces of Morgenthaler and Rosenfeld [19]:
Let $(n, \overline{n}) = (6, 26)$ or $(26, 6)$ and let X be a subset of E.
A point x of X is a *Morgenthaler's (simple) n-surface point* if:

1) $\#\mathcal{C}^x_n(|X|^x) = 1$; and
2) $\#\mathcal{C}^x_{\overline{n}}(|\overline{X}|^x) = 2$; we denote $\mathcal{C}^x_{\overline{n}}(|\overline{X}|^x) = \{C^{xx}, D^{xx}\}$; and
3) $\forall y \in N_n(x) \cap X$, $N_{\overline{n}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{n}}(y) \cap D^{xx} \neq \emptyset$.

If $\#\mathcal{C}^x_{\overline{n}}[N_{124}(x) \cap \overline{X}] = 2$, we say that the n-surface point x is *orientable*.
A *Morgenthaler's (simple) closed n-surface* is a finite n-connected set X consisting entirely of orientable Morgenthaler's n-surface points.

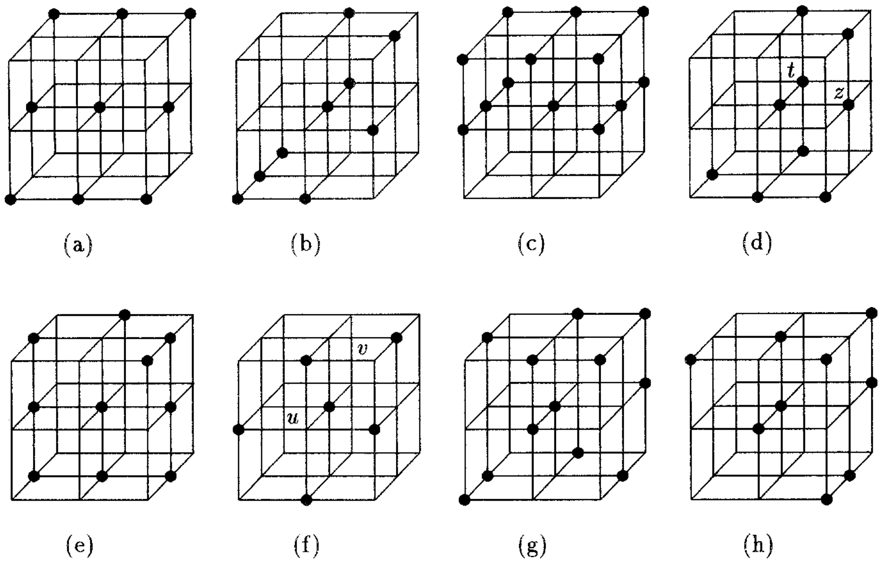

Fig. 3. Examples.

It was shown that [19]:

A Morgenthaler closed n-surface is a strongly n-separating set.

Furthermore, it was proved that the assumption of orientability is unnecessary for the 6-connectivity [21] and for the 26-connectivity [20].

Let X be a Morgenthaler's n-surface and $x \in X$. Since X is a strongly n-separating set, $\overline{X}$ has two components A and B and we may affirm that, up to a renaming of A and B, we have $C^{xx} \subset A$ and $D^{xx} \subset B$.

In Fig. 3, only the configurations (a) and (d) satisfy the conditions for Morgenthaler's 26-surfaces. As we have seen, the configuration (a) satisfies the conditions for strong 26-surfaces but the configuration (d) does not. Nevertheless, we will show that a Morgenthaler's 26-surface is necessarily a strong 26-surface. In fact, even if the configuration (d) satisfies the Morgenthaler's conditions, this configuration cannot belong to any Morgenthaler's surface. This happens since this configuration is not *extensible*: it is not possible that all the neighbors of the central point of this configuration satisfy the Morgenthaler's surface point conditions. This may be seen by considering the point z appearing in Fig. 3 (d): it is not possible that z satisfies the above condition 3, since the point t, which belongs to $N_{26}(z) \cap X$, has a single 6-neighbor in $\overline{X} \cap N_{26}(z)$. Then t cannot be $\overline{n}$-adjacent to both C^{zz} and D^{zz}.

The following result is a prerequisite for the main theorem of this section. It has been proved by enumerating with a computer all the configurations which are contained in a 26-neighborhood and which satisfy the Morgenthaler's conditions;

such an enumeration is within the reach of computers since there are at most 2^{26} cases to examine:

Lemma 8. *Let $x \in X$ be a Morgenthaler's 26-surface point. It is not possible that there is a 6-connected component in $|\overline{X}|^x$ which is not 6-adjacent to x, i.e., we have, up to a renaming of A and B, $C^{xx} = |A|^x$ and $D^{xx} = |B|^x$.*

The following proposition allows to characterize the homotopy of Morgenthaler's 26-surfaces (see [6] for the proof).

Theorem 9. *Any Morgenthaler's simple 26-surface is a strong 26-surface.*

On the other hand, it may be shown that (see [6]):
There are finite strong 26-surfaces which are not Morgenthaler's 26-surfaces.
Thus, though the conditions for strong surfaces may appear as restricting conditions, they are more general than the Morgenthaler's conditions for closed 26-surfaces.

6 Strong surfaces and Malgouyres' 18-surfaces

The definition of Malgouyres' surfaces is based on a generalization of the notion of a simple closed curve.
Let X be a subset of E. We say that a point x of X is an *n-corner* if x is n-adjacent to two and only two points y and z belonging to X such that y and z are themself n-adjacent; we say that the n-corner x is *simple* if y and z are not corners and if x is the only point n-adjacent to both y and z.
We say that X is a *generalized simple closed n-curve*, or a *G_n-curve*, if the set obtained by removing all simple n-corners of X is a simple closed n-curve.

Note that a G_n-curve is an elementary closed n-curve (see [16]). We present now the definition of a Malgouyres' 18-surface (see [15]):

A finite subset X of E is called a *Malgouyres' (simple) closed 18-surface* if X is 18-connected and if, for each x of X, the set $|X|^x$ is a G_{18}-curve.

In Figure 3, (a) (b), (e), (f) and (g) are examples of Malgouyres' surface points. It was proved in [15] that:
Any Malgouyres' 18-surface is a strongly n-separating set, for $n = 18$ and 26.
Let X be a Malgouyres' closed surface. Since a strongly separating set is thin, for each x of X, we have $\#C_{\overline{n}}^x(|\overline{X}|^x) = 2$, and we denote $C_{\overline{n}}^x(|\overline{X}|^x) = \{A^{xx}, B^{xx}\}$ with $A^{xx} \subset A$ and $B^{xx} \subset B$.

Our proof that a Malgouyres' 18-surface X is a strong 18-surface is based on the checking, for all possible configurations of G_{18}-curves contained in the neighborhood of a point x, of the conditions imposed by the definition of strong 18-surfaces (Th. 7). To do this, we need to check conditions dealing with $|A|^x$ and $|B|^x$ for $x \in X$. Contrary to the Morgenthaler's 26-surfaces (Lemma 8), the neighborhood of a Malgouyres' surface point may contain some connected components of $\overline{X}$ which are not adjacent to X (see Fig. 3 (f)). Therefore, we need to show that, for any x of X, the assignment of 6-connected components of $|\overline{X}|^x$ obtained by coloring with the same color any two points of the same 6-connected component of $\overline{X}$ is a strong assignment. For that purpose, we need to precise the structure of 6-connected components of $|\overline{X}|^x$. This is done with the following lemma (see [6] for the proof):

Lemma 10. *Let X be a finite Malgouyres' closed 18-surface. Let $x \in X$ and let C be a 6-connected component of $|\overline{X}|^x$ distinct from A^{xx} and B^{xx}. Then:*

1. C is a singleton; we denote $C = \{u\}$;

2. the point u is 6-adjacent to exactly 3 points y, z and t of $|X|^x$;

3. the points y, z and t are pairwise 18-neighbors;

4. the set $\{y, z, t\}$ contains a simple corner, say y, for $|X|^x$; the point y is 6-adjacent to exactly one element of $\{A^{xx}, B^{xx}\}$, say A^{xx} ;

5. u belongs to $|B|^x$.

Observe that point (5) provides a method for computing an assignment on X, using only the datum of the set $|X|^x$. This is required for an exhaustive checking of the conditions of Th. 7. Now we can set the main result of this section:

Theorem 11. *Any Malgouyres' closed 18-surface is a strong 18-surface.*

This result was proved automatically in the following way: For all possible extensible G_{18}-curves contained in the neighborhood of a point x, the two sets $|A|^x$ and $|B|^x$ were constructed according to Lemma 10, and the theorem 7 was checked.

Since we proved that any Morgenthaler's 26-surface is a Malgouyres' 18-surface [15], we have the corollary:

Any Morgenthaler's 26-surface is a strong 18-surface.

On the other hand, it may be shown that (see [6]):

- *there are Malgouyres' 18-surfaces which are not strong 26-surfaces;*
- *there are finite strong 26-surfaces which are not Malgouyres' 18-surfaces;*
- *there are finite strong 18-surfaces which are not Malgouyres' 18-surfaces.*

7 Conclusion

We have investigated the closed 26-surfaces defined by Morgenthaler and Rosenfeld, and the closed 18-surfaces defined by Malgouyres. We proved that these surfaces are strong surfaces, i.e, they satisfy a strong homotopy property: the closure of a back-component is strongly homotopic to that back-component. This means that we can homotopically remove any subset of a surface from the closure of a back-component. The interest of this result is that it provides a new general topological property of existing surfaces. Furthermore, strong surfaces appear as an intesting generalization of these surfaces.

This strong homotopic property has been obtained from a local characterization of P-simple points which may be viewed as the operating way of considering strong homotopy. When we transpose this characterization to the surfaces, we obtain a characterization which is "not fully" local. Our future work will include a fully local characterization of these surfaces as well as the Jordan's theorem required for this purpose.

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