The Quasi-Shear Rotation

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Abstract. A discrete one-to-one bitmap rotation called the Quasi Shear Rotation (QSR) is presented. This bitmap rotation is one-to-one, reversible and can have an arbitrary (non lattice) rotation center. The QSR represents so far the "best" choice of an one-to-one discrete rotation for a practical application.

1 Introduction

In the fields of Computer Graphics and Image Processing one of the most important and basic transformations is the rotation. Several different approaches have been used to rotate an image on a computer screen. One classical way to perform a bitmap rotation is to transform an image point per point by the Euclidean rotation and then to apply a discretization (coordinate approximation through a *round* function, color interpolation, etc.). Another method is the Paeth-Tanaka [5, 6, 8] approach that consists of decomposing a rotation matrix into three shear matrices. Each shear transformation is then easy to implement and interpolation of the pixel colors is straightforward. These classical methods are usually satisfying from the graphical point of view. Nevertheless, they imply a loss (as much as 17% for some angles) or/and irreversible transformation of the information (color information). To avoid loss or corruption of information, we propose in this paper a bitmap rotation called Quasi Shear Rotation (QSR) that is one-to-one, reversible, that can have an arbitrary center (not necessarily a lattice point) and that has good graphical properties.

To build our rotation we used an approach introduced by J-P. Réveillès [7]. We generalized his rotation by taking into account the rotation center and improved its graphical properties by using Paeth-Tanaka's rotation matrix decomposition. As Paeth and Tanaka did, we decompose a rotation matrix into three shear matrices, but we replace each of them by a quasi shear transformation. Each quasi shear transformation is a discrete one-to-one bitmap transformation that approximates a continuous shear transformation. The composition of these three quasi shear transformations forms the Quasi Shear Rotation (QSR). To evaluate the QSR and to design the final rotation algorithm we applied the *distance criterion* [1] to a 2048×2048 size image. The distance criterion works as follow : the average and maximal distance between the points rotated by the continuous rotation and the same points rotated by the QSR is computed. We achieve an average and maximum distance of $\simeq 0.45$ and $\simeq 1.10$ pixel units respectively. For a practical use by the computer graphics community, the QSR represents "best" discrete one-to-one rotation known so far.

In section 2 we introduce the quasi shear transformations and the Quasi Shear Rotation. Section 3 presents performance evaluation of the QSR and the final implementation. Final conclusion and discussion are given in section 4.

2 Quasi Shear Rotation

2.1 Preliminaries

In this paper we decompose the rotation matrix into three shear transformations as described by Paeth and Tanaka [5, 6, 8]:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & -\tan\frac{\theta}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \sin\theta & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\tan\frac{\theta}{2} \\ 0 & 1 \end{pmatrix} \tag{1}$$

Let us take an arbitrary real value $\omega > 0$ and let's pose $\alpha = \omega \sin\theta$, $\alpha' = \omega \sin\frac{\theta}{2}$ and $\beta' = \omega\cos\frac{\theta}{2}$. Then equation (1) can be written as:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\alpha'}{\beta'} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{\alpha}{\omega} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{\alpha'}{\beta'} \\ 0 & 1 \end{pmatrix} \tag{2}$$

Each of these shear transformations is a continuous transformation. We will now present discrete transformations that correspond to the continuous shear transformations in the discrete world. These are a special class of Quasi-Affine Transformations, called the Quasi Shear Transformations (QST). Many theoretical and practical results about the Quasi Affine Transformations are described in [3, 7].

Definition 1. Quasi Shear Transformations:
The Horizontal Quasi Shear is defined by :

$$\begin{aligned} HQS\,(a,b,c) \quad &: \quad \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} x + \left[\frac{ay+c}{b}\right] \\ y \end{pmatrix} \end{aligned}$$

The Vertical Quasi Shear is defined by :

$$\begin{aligned} VQS\,(a,b,c) \quad &: \quad \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} x \\ y + \left[\frac{ax+c}{b}\right] \end{pmatrix} \end{aligned}$$

with a, b, c integers, $b > 0$.

$\frac{a}{b}$ is called the *slope* of the quasi shear and c the *translation factor.* $[\cdot]$ is the notation for the *Integer Part Function invariant by Translation* (or *Floor* function). From a practical point of view the quasi shear transformations are very interesting because they correspond to row or column shifts that are easy to implement efficiently (in soft and hardware). It is obvious that the quasi shear transformations are one-to-one transformations. The Quasi Shear Transformations (QST) have been introduced by J-P. Réveillès to design his one-to-one bitmap rotation. The reader can find several other one-to-one discrete rotations in [1, 4].
Let's now derive a QST that is a *"good approximation"* of a continuous shear transformation, for a given ω. The following results are needed for the derivation:

Definition 2. Réveillès Discrete Line [7]:
A discrete line $L(a,b,c,\omega)$ is defined by :

$$\begin{aligned} L(a,b,c,\omega) &= \{(x,y) \in \mathbb{Z} \mid 0 \leq ax - by + c < \omega\} \\ &= \left\{(x,y) \in \mathbb{Z} \mid \left[\frac{ax - by + c}{\omega}\right] = 0\right\} \end{aligned} \tag{3}$$

with a, b, c, ω integers, $\omega > 0$.

$\frac{a}{b}$ is the *slope* of the discrete line, c is called the *translation factor* and ω is called the *arithmetical thickness.* The Réveillès line [7] is a generalization of the Bresenham line. The last result we need is the following :

Proposition 3. *Transformation of a discrete line :*

- *The* $HQS(a,b,c)$ *transforms the line* $L(1,0,-k,1)$ *(i.e.* $x = k$*) into the discrete line* $L(-b,-a,c+bk,b)$ *.*
- *The* $VQS(a,b,c)$ *transforms the line* $L(0,-1,-k,1)$ *(i.e.* $y = k$*) into the discrete line* $L(a,b,c+bk,b)$ *.*

This result is a immediate extension of a similar result presented with $c = 0$ by J-P. Réveillès in [7]. The proof is obvious. Let us now suppose that we want to find a discrete QST that approximates the following continuous shear transformation :

$$\begin{aligned} CHS : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{cases} x' = x + \frac{a}{b}(y - y_o) \\ y' = y \end{cases} \end{aligned}$$

with a, b integers and $b > 0$. Obviously the slope of the quasi shear transformation will be $\frac{a}{b}$.

Note that this shear transformation leaves the points of the line $y = y_o$ unchanged. If we use this horizontal shear, a vertical shear that leaves the line $x = x_o$ unchanged and apply expression (1) we have a continuous rotation of center (x_o, y_o). The idea behind the Quasi-Shear rotation is simple. If we find the quasi-shear transformations that best approximate the horizontal and vertical shear that form the continuous rotation of center (x_o, y_o) then we have a discrete rotation that approximates the continuous rotation of center (x_o, y_o).

What we have to find is the translation constant c. CHS applied to the line $x = k$ transforms it into the line $-bx + ay + bk - ay_o = 0$. The corresponding discrete line that best approximates this line is : $-\frac{b}{2} \leq -bx+ay+bk-ay_o < \frac{b}{2}$ or $L\left(-b, -a, bk + \left[\frac{b-2ay_o}{2}\right], b\right)$. Therefore, considering the results of proposition 3, the translation constant we seek is $c = \left[\frac{b-2ay_o}{2}\right]$.

One last question remains open: what is the best choice for ω, and subsequently for a, a' and b'? Different approaches have been tested, however none of them seems to lead to significantly better results. Therefore we chose the most straightforward approach :

$$\begin{aligned} \omega &= 2^{15} \\ a &= [\omega \cdot \cos\theta] \\ a' &= \left[\omega \cdot \cos\frac{\theta}{2}\right] \text{ and } b' = \left[\omega \cdot \sin\frac{\theta}{2}\right] \end{aligned} \tag{4}$$

This choice of ω avoids some divisions, replaced by bit-shift operations. Depending on the size of the image ω can be larger or smaller but experimental results suggest (see section 3) that $\omega = 32768 = 2^{15}$ for an image of size 2048×2048 is large enough to ensure a discrete rotation of satisfactory quality.

2.2 Definition and properties

With help of the results obtained in the preliminaries, we can now design the following transformation called Quasi Shear Rotation :

Definition 4. Quasi Shear Rotation:
A Quasi Shear Rotation of center (x_o, y_o) is defined by :

$$\begin{aligned} QSR(\theta, x_o, y_o) \quad : \quad & \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto HQS' \circ VQS' \circ HQS' \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

with:

$$\begin{aligned} HQS' &= HQS\left(-a', b', \left[\frac{b' - 2y_o a'}{2}\right]\right) \\ VQS' &= VQS\left(a, \omega, \left[\frac{\omega - 2ax_o}{2}\right]\right) \end{aligned}$$

and θ, x_o, y_o real; a, a', b', ω integers verifying (4).

To conclude the following proposition, whose proof are trivial, are presented:

Proposition 5. *The QSR is a one-to-one bitmap transformation.*

Proposition 6. *The QSR is reversible and*
$QSR(\theta, x_o, y_o)^{-1} = QSR(-\theta, x_o, y_o)$.

3 Evaluation of the Rotation and Algorithm

In order to evaluate the quality of the rotation we applied the *distance criterion* [1] to an image of size $[0, 1023] \times [0, 1023]$ for all the angles $\theta = \frac{k\pi}{4096}$ for $0 \leq k \leq 8192$, so $0 \leq \theta \leq 2\pi$. The rotation center is considered to be in $\left[-\frac{1}{2}, \frac{1}{2}\right[\times \left[-\frac{1}{2}, \frac{1}{2}\right[$. Considering the symmetries, this is as if we applied the criterion to an image of size 2048×2048 with a rotation origin in the center of it. Euclidean and discrete rotations are applied to each point in the image. The distance between the two resulting points are computed. The average and the maximum values of these distances are computed for each angle θ, and plotted in Fig.1 to 3. The distances are given in pixel units. The smaller the distance the better a discrete rotation approximates the Euclidean rotation from the distance point of view. This criterion allows a good evaluation of the graphical behavior and gives a good indication of the distortion that might appear in the final image.
For the classical truncated rotation algorithm (euclidean rotation composed with round function) the average distance lies around $\frac{1}{12}\left(2\sqrt{2} + \ln\left(3 + 2\sqrt{2}\right)\right)$ $\simeq 0.3826...$ and the maximal distance is less than $\frac{\sqrt{2}}{2} \simeq 0.7071...$ [1].

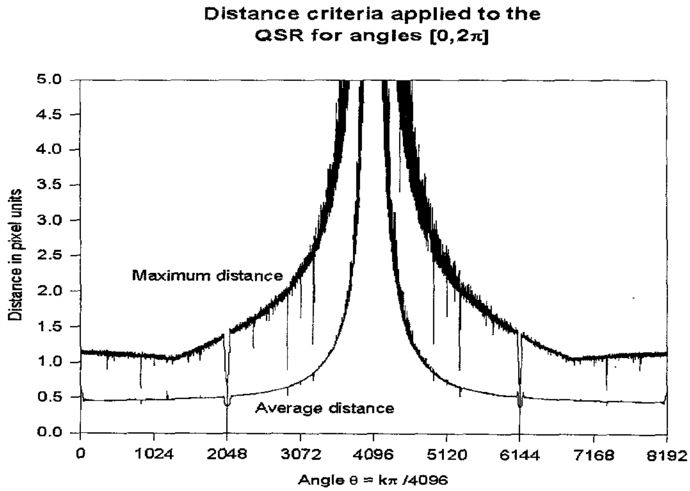

Figure 1: Quasi Shear Rotation with center (0,0)

Comments on Figure 1:

– Proposition 6 explains why the results obtained have a symmetry in

e value of $\simeq 0.4077..$ and $\simeq 0.6375..$ as average and maximum distance (see also Fig. 2).

– We performed the same computations with $\omega = 2^{19} = 524288$ and compares them with those obtained for $\omega = 2^{15}$. The gain in average distance is less then 0.01 pixel units and the maximal distance gain is less than 0.1 pixel units. On the other side, by taking $\omega = 2^n < 2^{15}$ the distances increase rapidly. It seems that $\omega = 2^{15}$ is a good practical choice that limits overflow problems and allows efficient implementation possibilities.

– In Figure 1 the criterion shows "good" graphical results for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and for values of θ close to $\frac{\pi}{2}$. On the other hand the results are not "good" for θ tending toward π. If we want to have a rotation algorithm that works for all angles we need to expand the results obtained for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ to all angles (see Figure 2). This is achieved by the following algorithm :

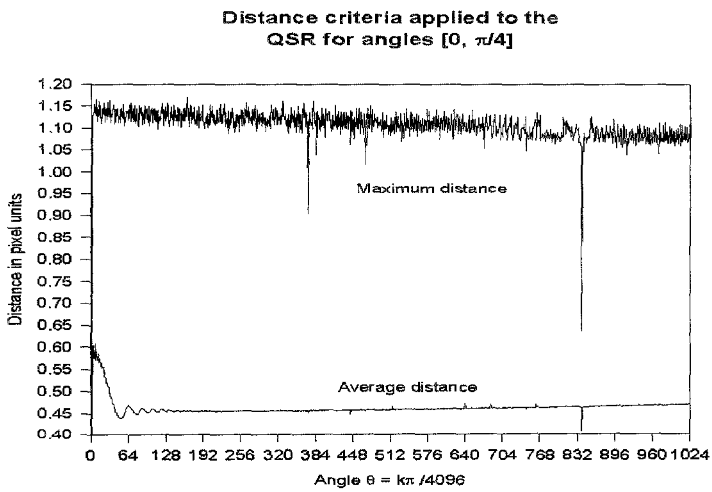

Figure 2: The best results for the QSR are obtained with the angles $\left[0, \frac{\pi}{4}\right]$

Let us introduce the algorithm of the Final QSR:
Algorithm $FINAL\_QSR\,(\theta, x_o, y_o, Image\_init, Image\_final)$

- $\theta_f = \theta - 2\pi \cdot \left\lfloor \frac{\theta + \frac{\pi}{4}}{2\pi} \right\rfloor$
- $quadrant = \left\lfloor \frac{1}{2} + \frac{2\theta_f}{\pi} \right\rfloor$
- $\theta_f = \theta_f - quadrant \cdot \frac{\pi}{2}$
- $case\ (quadrant):$
  - 0 : Apply $QSR\,(\theta, x_o, y_o)$ to all points of $Image\_init$ and write to $Image\_final$. break.
  - 1 : Apply $QSR\,(\theta_f, x_o, y_o)$ and then $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to all points of $Image\_init$ and write to $Image\_final$. break.
  - 2 : Apply $QSR\,(\theta_f, x_o, y_o)$ and then $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to all points of $Image\_init$ and write to $Image\_final$. break.
  - 3 : Apply $QSR\,(\theta_f, x_o, y_o)$ and then $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to all points of $Image\_init$ and write to $Image\_final$. break.
- End Algorithm.

Figure 3 presents the final distance criterion results obtained for the QSR rotation as described by the algorithm. Figure 4 shows the different steps of a QSR rotation. Figure 5 presents the results obtained after a rotation of an angle of $\frac{\pi}{4}$.

4 Conclusion

In this paper we propose a one-to-one bitmap rotation based on the decomposition of the rotation matrix into three shear matrices. The resulting Quasi shear Rotation is one-to-one, reversible, easy to implement and provides a "good" approximation of the continuous rotation. The rotation center can be any arbitrary point. The average and maximum distance between a point rotated by the QSR and by the continuous rotation, for a 2048×2048 size image, are of $\simeq 0.45$ and $\simeq 1.10$ pixel units, respectively. The proposed algorithm achieves a subpixel one-to-one discrete approximation of the continuous rotation. Only the Pythagorean one-to-one discrete rotation has better results than those of the QSR but only for some angles. For that reason, the QSR seems to be the best suited one-to-one discrete rotation for practical applications. This study of the Quasi Shear Rotation and the study of the Pythagorean Rotation [1, 4] suggests that there is an underlying arithmetical structure that has not yet been fully uncovered. There is a link between the Pythagorean triplets, the continuous rotation, the discrete line and the different discrete rotations that has only partially been studied so far. Understanding the theoretical aspects of this link should open the way to even better one-to-one discrete rotations in the future.

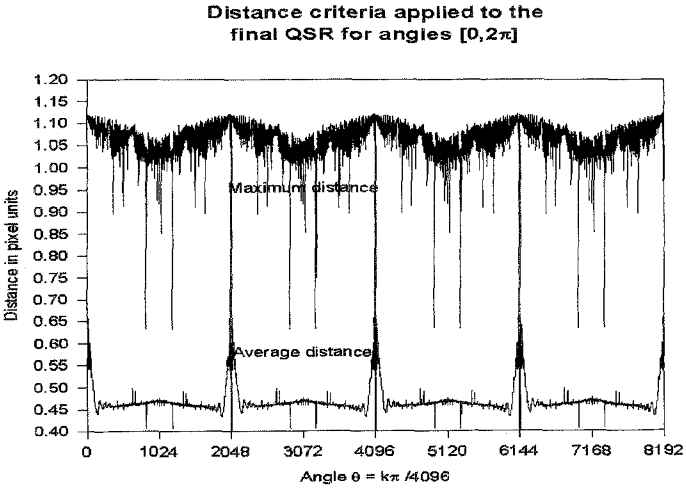

Figure 3: Distance criteria results for Final QSR

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