Reconstructing Convex Polyominoes from Horizontal and Vertical Projections II

Elena Barcucci[1], Alberto Del Lungo[1], Maurice Nivat[2], Renzo Pinzani[1]

[1] Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, Firenze, Italy, pire@ingfil.ing.unifi.it
[2] LITP Institut Blaise Pascal, Université Paris 7 "Denis Diderot", 2 Place Jussieu, Paris Cedex 05, France, Nivat@litp.ibp.fr

Abstract. In [1], we studied the problem of reconstructing a discrete set S from its horizontal and vertical projections. We defined an algorithm that establishes the existence of a convex polyomino Λ whose horizontal and vertical projections are equal to a pair of assigned vectors (H, V), with $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$. Its computational cost is $O(n^4m^4)$. In this paper, we introduce some operations for recontructing convex polyominoes by means of vectors H's and V's partial sums. These operations allows us to define a new algorithm whose complexity is less than $O(n^2m^2)$.

1 Introduction

A *cell* is a unitary square $[i, i+1] \times [j, j+1]$ in which $i, j \in \mathbb{N}_0$. Let S be a finite set of cells. A *column* (*row*) of S is the intersection of S with an infinite vertical strip $[i, i+1] \times \mathbb{R}$ (horizontal $\mathbb{R} \times [i, i+1]$) in which $i \in \mathbb{N}_0$. The i-th *row projection* and the j-th *column projection* of S are the number of cells in S's i-th row and j-th column, respectively. We dealt with the reconstruction of objects from their projections: with regard to establishing the existence of an S set of cells in which the i-th row projection and the j-th column projection are equal to h_i and v_j, respectively, and $H = (h_1, h_2, \ldots, h_m) \in \mathbb{N}^m$ and $V = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ are two assigned vectors. In [1], we studied the problem with respect to some cell set classes on which we imposed some connection constraints and devised an algorithm for convex polyomino reconstruction. This algorithm establishes the existence of a convex polyomino Λ having projections equal to (H, V). Moreover, if there is at least one convex polyomino having projection (H, V), the algorithm reconstructs one of them in a maximum of $O(n^4m^4)$ time. In this paper, we deduce some operations (called *partial sum operations*) for the reconstruction of convex polyominoes, from some properties of H and V's partial sums. We use these operations to define a new algorithm in which it is not necessary to find out feet's positions, whereas the "old" algorithm has to examine all of them (i.e. $O(n^2m^2)$ positions). Since the computational cost of a partial sum operations is $O(n\ m)$, new algorithm's complexity is less than $O(n^2m^2)$ and is therefore smaller than the previous algorithm's. At the moment, however, we only have experimental evidence to support the fact that our algorithm establishes the existence of a convex polyomino A whose projections are equal to (H, V), for all instances (H, V). We wish to point out that Woeginger [3] proved that the reconstruction problem in the classes of horizontally and vertically convex sets (**h**,**v**) and polyominoes (**p**) is an NP-complete problem. In [1], we showed that the reconstruction is NP-complete in the classes (**p**,**h**), (**p**,**v**), (**h**) and (**v**). Therefore, the problem can be solved in polynomial time only if all three properties **p**, **h** and **v**, are verified by the cell set. This, in turn, means that the set is a convex polyomino.

2 Preliminaries

A *polyomino* is a connected finite set of adjacent cells lying two by two along a side and it is defined up to a translation. A polyomino is *convex* if all its columns and rows are connected. We denote the class of convex polyominoes by (**p**,**h**,**v**). Let $H = (h_1, h_2, \ldots, h_m) \in \mathbb{N}^m$ and $V = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$. The pair (H, V) is said to be *satisfiable* in class (**p**,**h**,**v**) if there is at least one convex polyomino S such that S's i-th row projection and j-th column projection (starting from the upper-left corner) are equal to h_i and v_j, respectively, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. We also say that S satisfies (H, V) in (**p**,**h**,**v**). From the definition, we can deduce that if (H, V) is satisfiable in (**p**,**h**,**v**), then:

$$\forall i \in [1..m]\ 1 \leq h_i \leq n, \quad \forall j \in [1..n]\ 1 \leq v_j \leq m, \quad \sum_{j=1}^{m} h_j = \sum_{i=1}^{n} v_i. \tag{2.1}$$

Consequently, if a convex polyomino S satisfies a pair (H, V) having $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$, S is contained in a rectangle R of size $n \times m$ (see fig. 1).

3 Some convex polyomino properties

Let us take two vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$ and a convex polyomino A that satisfies (H, V). A is contained in a rectangle R of size $n \times m$. Let $[S, S']$ ($[N, N']$, $[E, E']$, $[W, W']$) be the intersection of A's boundary on R's lower (upper, right, left) side. Segment $[S, S']$ is the base of a set made up of h_m consecutive columns of A, called A's *foot*, denoted as P_S. In the same way, we define A's other three feet P_N, P_E and P_W by referring to intersections $[N, N']$, $[E, E']$, $[W, W']$. We denote feet P_S's, P_N's, P_E's and P_W's by $[s_1..s_2]$, $[n_1..n_2]$, $[e_1..e_2]$ and $[w_1..w_2]$, respectively. Let $c(u_j, j)$ and $c(d_j, j)$ be the upmost and lowest cells of A's j-th column (see fig. 2). Let W_j be the set of columns denoted by A_k, with $k \in [1..j]$, and let and N_i be the set of rows denoted by A^k, with $k \in [1..i]$.

Proposition 1. *i)* $N_{u_j-1} \subset W_{j-1}$, *for* $n_2 < j \leq n$,
ii) $W_j \subset N_{d_j}$, *for* $1 \leq j < s_1$,
iii) $N_{u_j-1} \subset A - W_j$, *for* $1 \leq j < n_1$,
iv) $A - W_{j-1} \subset N_{d_j}$, *for* $s_2 < j \leq n$.

Proof. (i) Let $c(i, k) \in N_{u_j-1}$, with $n_2 < j \leq n$. We have to prove that $k < j$. As we proceed from left to right starting from n_2-th column, the ordinates of the columns' upmost cells increase progressively. Therefore, if $k \geq j$, upmost

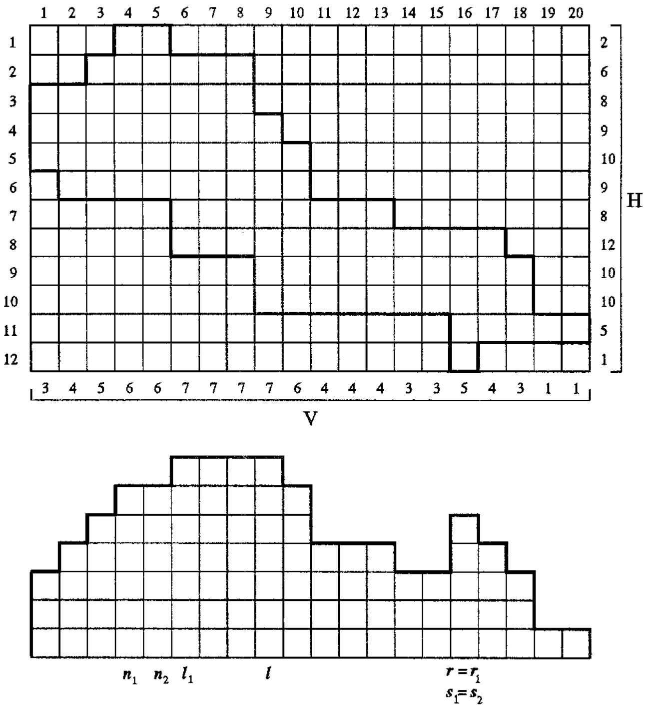

Fig. 1. A convex polyomino that satisfies (H, V) and its vertical projection

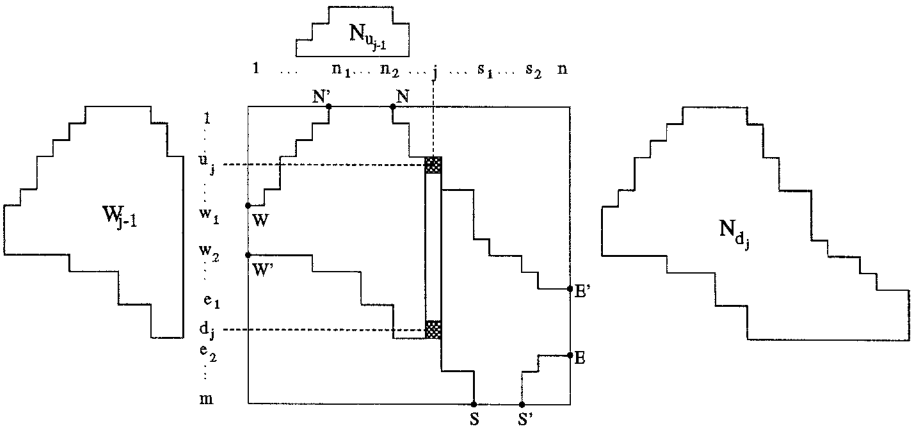

Fig. 2. A convex polyomino's feet

$c(u_k, k)$ and $c(u_j, j)$ cells' ordinates are such that $u_k \geq u_j$. This is impossible because $c(u_k, k)$ is the k-th column's upmost cell and so $u_k \leq i$ and we obtain $i < u_j$ from $c(i, k) \in N_{u_j - 1}$. Therefore, we get $k < j$.
In the same way we can prove the properties (ii), (iii) and (iv).

Let $H_k = \sum_{j=1}^{k} h_j$, $V_k = \sum_{i=1}^{k} v_i$ and $A = \sum_{j=1}^{m} h_j = \sum_{i=1}^{n} v_i$. From the previous Proposition we get:

Corollary 2. *i)* $H_{u_j-1} < V_{j-1}$, *for* $n_2 < j \leq n$,
ii) $V_j < H_{d_j}$, *for* $1 \leq j < s_1$,
iii) $H_{u_j-1} < A - V_j$, *for* $1 \leq j < n_1$,
iv) $A - V_{j-1} < H_{d_j}$, *for* $s_2 < j \leq n$.

We can deduce the same properties for the rows.

3.1 The feet's positions

From the definition of convex polyomino, it follows that two pairs of consecutive feet have a non-empty intersection. In other words,
$(P_N \cap P_W \neq \emptyset$ and $P_S \cap P_E \neq \emptyset)$ or $(P_N \cap P_E \neq \emptyset$ and $P_S \cap P_W \neq \emptyset)$ (3.2)
Let us now assume that $v_j < m$ for all $j \in [1..n]$. Then, $P_N \cap P_S = \emptyset$. Moreover, from A's convexity, we deduce that lengths $v_{n_1}, v_{n_1+1}, \ldots, v_{n_2}$ and $v_{s_1}, v_{s_1+1}, \ldots, v_{s_2}$ of P_N's and P_S's columns are such that:
- if P_N is to the left of P_S $(n_2 < s_1)$, then $v_{n_1} \leq v_{n_1+1} \leq \cdots \leq v_{n_2}$ and $v_{s_1} \geq v_{s_1+1} \geq \cdots \geq v_{s_2}$ (see fig. 1),
- if P_N is to the right of P_S $(s_2 < n_1)$, then $v_{s_1} \leq v_{s_1+1} \leq \cdots \leq v_{s_2}$ and $v_{n_1} \geq v_{n_1+1} \geq \cdots \geq v_{n_2}$.
Let l and r be $l = \max\{j \in [1..n] : v_q \leq v_{q+1}, \forall q \in [1..j-1]\}$, and $r = \min\{j \in [1..n] : v_q \geq v_{q+1}, \forall q \in [j..n-1]\}$ (see fig. 1). If P_N is not contained in A's first l columns or last $n - r - 1$ columns, then we have the disconnection shown in fig. 3(a). Therefore, from the convexity property we obtain:

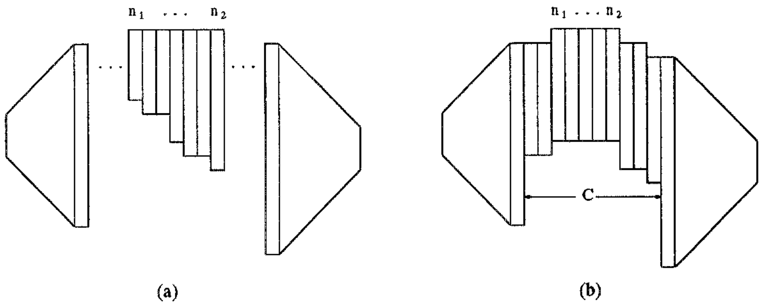

Fig. 3. Two illegal positions of A's foot P_N

Proposition 3. *If there is a convex polyomino A that satisfies (H, V) with $v_j < m$ for all $j \in [1..n]$, then the positions $[n_1..n_2]$ and $[s_1..s_2]$ of A's feet P_N and P_S are such that: $(n_2 \leq l$ and $s_1 \geq r)$ or $(s_2 \leq l$ and $n_1 \geq r)$.*

Let us now assume that there is a set C of adjacent columns having the same length (always less than m), with $C \subseteq W_l$ (or $A - W_{r-1}$). If C's columns are more than the number h_1 of P_N's columns and $P_N \cap C \neq \emptyset$, then $P_N \cap C$ is contained in C's first or last h_1 columns. If this condition does not occur, then we obtain the disconnection illustrated in fig. 3(b). Therefore, if we denote l_1 and r_1 as $l_1 = \min\{j \in [1..n] : v_j = v_l\}$, $r_1 = \max\{j \in [1..n] : v_j = v_r\}$, (see fig. 1) being $P_N \subseteq W_l$ and $l_1 + h_1 - 1 < l$, we can deduce that $P_N \subseteq W_{l_1+h_1-1}$.

Consequently, by setting: $l_N = \min\{l_1 + h_1 - 1, l\}$, $l_S = \max\{l_1 + h_m - 1, l\}$, $r_S = \max\{r_1 + h_m - 1, r\}$, and $r_N = \max\{r_1 + h_1 - 1, r\}$, we obtain:

Proposition 4. *If a convex polyomino Λ exists that satisfies (H, V) with $v_j < m$ for all $j \in [1..n]$, then positions $[n_1..n_2]$ and $[s_1..s_2]$ of Λ's feet P_N and P_S are such that: $(n_2 \leq l_N$ and $s_1 \geq r_S)$ or $(s_2 \leq l_S$ and $n_1 \geq r_N)$.*

We now examine the case in which there is at least one $j \in [1..n]$ such that $v_j = m$: the j-th column belongs to both feet P_N and P_S. It follows from Λ's convexity that if there is a set M of m-long columns, these columns are adjacents and are contained in P_N and P_S. Moreover, V's elements are a unimodal sequence and we have the three cases are illustrated, in fig. 4 (a), (b) and (c):

Proposition 5. *The numbers h_1 and h_m of P_N and P_S's columns are such that $h_1 \geq l - r + 1$ and $h_m \geq l - r + 1$. Moreover,*
a) if $h_1 > l - r + 1$ and $h_m > l - r + 1$, then: $n_1 = l - h_1 + 1$, $n_2 = l$ and $s_1 = r$, $s_2 = r + h_m - 1$, or $n_1 = r$, $n_2 = r + h_m - 1$ and $s_1 = l - h_1 + 1$, $s_2 = l$.
b) If $h_1 = l - r + 1$ and $h_m > l - r + 1$, then: $n_1 = r$, $n_2 = l$ and $s_1 \geq l - h_m + 1$, $s_2 \leq r + h_m - 1$.
c) If $h_1 > l - r + 1$ and $h_m = l - r + 1$, then: $n_1 \geq l - h_1 + 1, n_2 \leq r + h_m - 1$ and $s_1 = r$, $s_2 = l$.

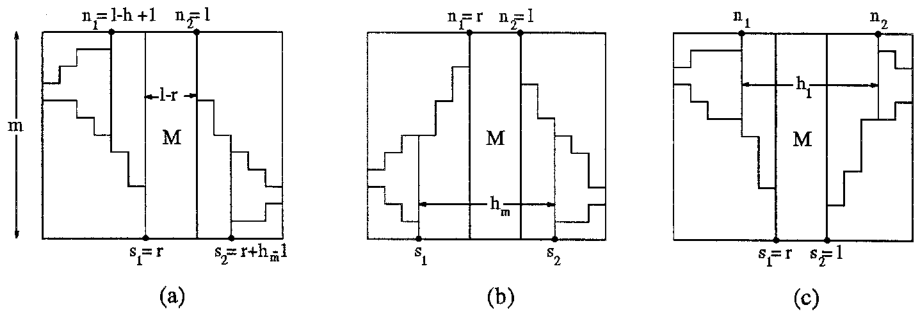

Fig. 4. Some convex polyominoes containing a set of m-long columns

In the same way, by using the vector H, we can deduce that feet P_W's and P_E's position have analogous limitations.

4 Partial sum operations

In this section, by using the properties of partial sums H_i, V_j and the feet's positions, we define some operations for reconstructing convex polyominoes Λ from their projections (H, V). We call any set α of cells such that $\alpha \subseteq \Lambda$ a *kernel*, and we call any set β of cells such that $\Lambda \subseteq \beta \subseteq R$ a *shell* , where R is the rectangle containing Λ. Assuming that $\alpha := \emptyset$ and $\beta := R$, we define the *partial sum operations* for Λ's reconstruction that reduce the shell and expand the kernel. We reduce the shell by eliminating the cells not belonging to Λ from

β. Vice versa, we expand the kernel by putting the cells belonging to Λ into α. We label α's cells "1" and the ones not belonging to β "0".

Let us take two vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$. From Corollary 2's conditions we can deduce a lowest (upmost) bound of the upmost (lowest) cells of Λ's columns, where Λ satisfies (H, V). Let $U_j^{(1)} = \max i$ such that $H_{i-1} < A - V_j$, $U_j^{(2)} = \max i$ such that $H_{i-1} < V_{j-1}$ where $i \in [2..m+1]$, and $D_j^{(1)} = \min i$ such that $A - V_{j-1} < H_i$, $D_j^{(2)} = \min i$ such that $V_j < H_i$ where $i \in [1..m]$.
If we know the position n_1 and n_2 of Λ's foot P_N, from conditions (i) and (iii) of Corollary 2, the ordinate u_j of the upmost cell $c(u_j, j)$ is:
$u_j \le U_j^{(1)}$, for $1 \le j < n_1$ and $u_j \le U_j^{(2)}$, for $n_2 < j \le n$.
Likewise, if we know the positions s_1 and s_2 of Λ's foot P_S, then from conditions (ii) and (iv) of Corollary 2, we deduce that the ordinate d_j of the lowest cell $c(d_j, j)$ is: $d_j \ge D_j^{(1)}$, for $1 \le j < s_1$ and $d_j \ge D_j^{(2)}$, for $s_2 < j \le n$.
Therefore, $U_j^{(p)}$ with $p = 1, 2$, is the lowest bound of u_j, while $D_j^{(q)}$ with $q = 1, 2$, is the upmost bound of d_j. For instance, some positions of $U_j^{(2)}$ and $D_j^{(1)}$ are illustrated in fig. 5. Since the length of the j-th column is v_j, we obtain the following shell reduction from these bounds:
1) if $1 \le j < n_1$ and $i \ge U_j^{(1)} + v_j$, then $c(i, j) \notin \Lambda$;
2) if $n_2 < j \le n$ and $i \ge U_j^{(2)} + v_j$, then $c(i, j) \notin \Lambda$;
3) if $1 \le j < s_1$ and $i \le D_j^{(1)} - v_j$, then $c(i, j) \notin \Lambda$;
4) if $s_2 < j \le n$ and $i \le D_j^{(2)} - v_j$, then $c(i, j) \notin \Lambda$.
We now consider a column in which there are both bounds $U_j^{(p)}$ and $D_j^{(q)}$. Moreover, we assume that this column's bounds are such that: $U_j^{(p)} \le D_j^{(q)}$.
If $D_j^{(q)} - U_j^{(p)} + 1 > v_j$, then $d_j - u_j > v_j$. Consequently, there is no a convex polyomino Λ that satisfies (H, V).
If $D_j^{(q)} - U_j^{(p)} + 1 \le v_j$, then we get an expansion of the kernel, that is $c(i, j) \in \Lambda$, for $U_j^{(p)} \le i \le D_j^{(q)}$, because $U_j^{(p)}$ is the lowest bound of $c(u_j, j)$'s position and, therefore, we have $i_1 \le U_j^{(p)}$ such that $c(i_1, j) \in \Lambda$. Likewise, since $D_j^{(q)}$ is the upmost bound of $c(d_j, j)$'s position, we obtain $i_2 \ge D_j^{(q)}$, such that $c(i_2, j) \in \Lambda$. By means of Λ's convexity, we deduce that $c(i, j) \in \Lambda$, for $i_1 \le i \le i_2$ and $U_j^{(p)} \le i \le D_j^{(q)}$. Therefore:
5) if j exists such that $D_j^{(q)} - U_j^{(p)} + 1 > v_j$, there is no convex polyomino that satisfies (H, V); otherwise, if $1 \le D_j^{(q)} - U_j^{(p)} + 1 \le v_j$, then $c(i, j) \in \Lambda$, for $U_j^{(p)} \le i \le D_j^{(q)}$.
It is worth noting that if $D_j^{(q)} < U_j^{(p)}$, we cannot expand the kernel, but we can reduce the shell by means of steps (1)-(4). Steps (1)-(5) are called partial sum operations. Unfortunately, we do not usually know the feet's positions and so if we want to perform any partial sum operation, we have to use the properties of the feet's positions as determined in the previous section. We start out by

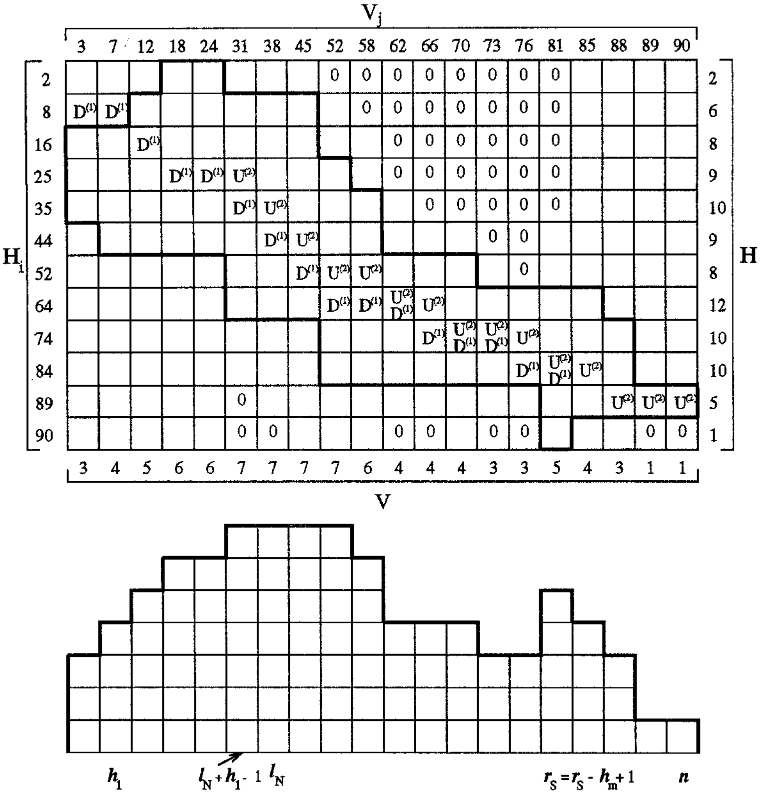

Fig. 5. Bounds $U_j^{(2)}$, $D_j^{(1)}$ and the shell reduction obtained by operations (2) and (3)

assuming that $v_j < m$ for all $j \in [1..n]$. From Proposition 4, we get the following two cases:

a) P_N is contained in the first l_N columns and P_S in the last $n - r_S - 1$ columns;

b) P_S is contained in the first l_S columns and P_N in the last $n - r_N - 1$ columns.

In the first case, we have $n_2 \leq l_N$ and $s_1 \geq r_S$. Consequently, by determining $U_j^{(2)}$ for $l_N < j \leq n$, and $D_j^{(1)}$ for $1 \leq j < r_S$, we get the lowest bound of u_j (i.e. $u_j \leq U_j^{(2)}$) and the upmost bound for d_j (i.e. $d_j \geq D_j^{(1)}$). Foot P_N is made up of h_1 columns and so j-th column, with $l_N - h_1 + 1 \leq j \leq l_N$, belongs to P_N or is on its right. Therefore, $u_j = 0$ or $u_j \leq U_j^{(2)}$, and so $U_j^{(2)}$ is the lowest bound of u_j, for $l_N - h_1 + 1 \leq j \leq n$. Analogously, $D_j^{(1)}$ is the upmost bound of d_j, for $1 \leq j \leq r_S + h_m - 1$. Consequently, we can perform partial sums operations (2) and (3) as follows:

a.2) if $l_N - h_1 + 1 \leq j \leq n$ and $i \geq U_j^{(2)} + v_j$, then $c(i,j) \notin \Lambda$;

a.3) if $1 \leq j \leq r_S + h_m - 1$ and $i \leq D_j^{(1)} - v_j$, then $c(i,j) \notin \Lambda$.

Bounds $U_j^{(2)}$, $D_j^{(1)}$ and the consequent shell's reduction are illustrated in fig. 5. Let us now consider the j-th columns, with $l_N - h_1 + 1 \leq j \leq r_S + h_m - 1$ (i.e., the columns having both bounds $U_j^{(2)}$ and $D_j^{(1)}$). By means of $U_j^{(2)}$'s and

$D_j^{(1)}$'s definitions, we get: $H_{U_j^{(2)}-1} < V_{j-1} < V_j < H_{D_j^{(1)}}$, and consequently: $U_j^{(2)} \le D_j^{(1)}$. We then perform the partial sums operation (5) as follows:
a.5) if there is $j \in [l_N - h_1 + 1..r_S + h_m - 1]$ such that $D_j^{(1)} - U_j^{(2)} + 1 > v_j$, then there is no convex polyomino that satisfies (H, V); otherwise, if $D_j^{(1)} - U_j^{(2)} + 1 \le v_j$ for all $j \in [l_N - h_1 + 1..r_S + h_m - 1]$, then $c(i,j) \in \Lambda$ for $U_j^{(2)} \le i \le D_j^{(1)}$.
This kernel's expansion is shown in fig. 6, and cells $c(i,j)$ with $j \in [l_N - h_1 + 1..r_S + h_m - 1]$ belong to Λ.

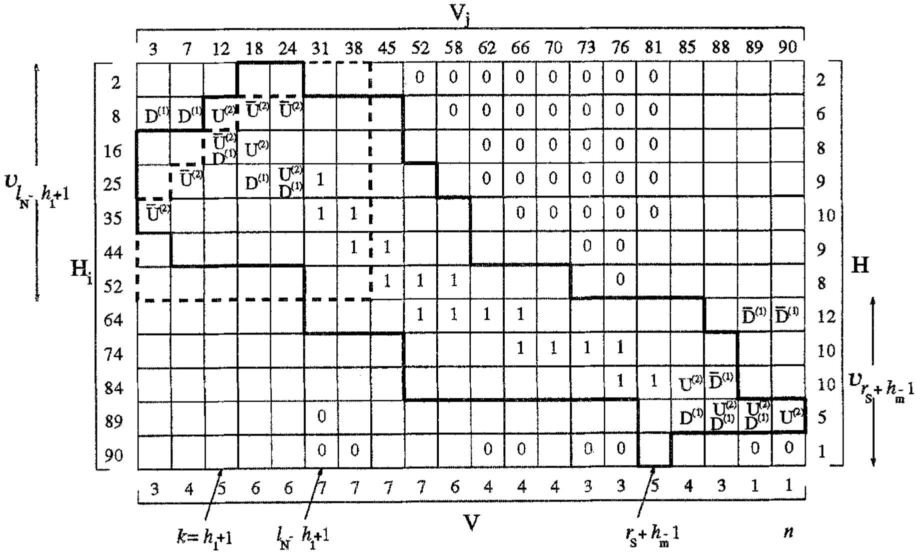

Fig. 6. The expansion of kernel obtained by operation (5)

Let us now consider the j-th column, with $h_1 + 1 \le j \le l_N - h_1$. It is either to the right or left of P_N, or belongs to it. If it is to the right of P_N or belongs to it, then $u_j \le U_j^{(2)}$ or $u_j = 0$. If the j-th column is to the left of P_N, then we obtain the column's lowest position when P_N is in the rightmost position (i.e., $[n_1..n_2] = [l_N - h_1 + 1..l_N]$). From Λ's convexity we deduce that the ordinate d_j of the j-th column's lowest cell cannot be greater than $v_{l_N - h_1 + 1}$ (see fig. 6). As a consequence, $u_j \le \bar{U}_j^{(2)}$, where $\bar{U}_j^{(2)} = v_{l_N - h_1 + 1} - v_j + 1$. Therefore, if $h_1 + 1 \le j \le l_N - h_1$, then $\max\{U_j^{(2)}, \bar{U}_j^{(2)}\}$ is the lowest bound of u_j. By perfoming partial sum operation (5), we obtain that, if $\max\{U_j^{(2)}, \bar{U}_j^{(2)}\} \le D_j^{(1)}$, the cells between $D_j^{(1)}$ and $\max\{U_j^{(2)}, \bar{U}_j^{(2)}\}$ belong to Λ, while, if $D_j^{(1)} < \max\{U_j^{(2)}, \bar{U}_j^{(2)}\}$, (because $U_j^{(2)} \le D_j^{(1)}$) we obtain $\bar{U}_j^{(2)} = \max\{U_j^{(2)}, \bar{U}_j^{(2)}\}$. In this case, the ordinate d_j cannot be greater than $v_{l_N - h_1 + 1}$, that is, $c(i,j) \notin \Lambda$ for $i > v_{l_N - h_1 + 1}$. Sequences $\{U_{h_1+1}^{(2)}, U_{h_1+2}^{(2)}, \dots, U_{l_N - h_1}^{(2)}\}$ and $\{D_{h_1+1}^{(1)}, D_{h_1+2}^{(1)}, \dots, D_{l_N - h_1}^{(1)}\}$ are two increasing sequences with $D_j^{(1)} \ge U_j^{(2)}$ for each j, while $\{\bar{U}_{h_1+1}^{(2)}, \bar{U}_{h_1+2}^{(2)}, \dots, \bar{U}_{l_N - h_1}^{(2)}\}$ is a decreasing sequence. Hence, there is a $k \in [(h_1 + 1)..h_{l_N - h_1}]$ such that:

$U_j^{(2)} \leq D_j^{(1)} < \bar{U}_j^{(2)}, j \in [(h_1+1)..(k-1)]$, $\max\{U_j^{(2)}, \bar{U}_j^{(2)}\} \leq D_j^{(1)}, j \in [k..h_{l_N-h_1}]$, (see fig. 6) and by performing operations (2) and (5), we obtain the following shell reduction and kernel expansion:
a.2) $c(i,j) \notin \Lambda$, for $i > v_{l_N-h_1+1}$ and $j \in [(h_1+1)..(k-1)]$;
a.5) $c(i,j) \in \Lambda$, for $\max\{U_j^{(2)}, \bar{U}_j^{(2)}\} \leq i \leq D_j^{(1)}$ and $j \in [k..h_{l_N-h_1}]$;
(see fig. 7). Finally, we consider the j-th column, $1 \leq j \leq h_1$, which is ei-

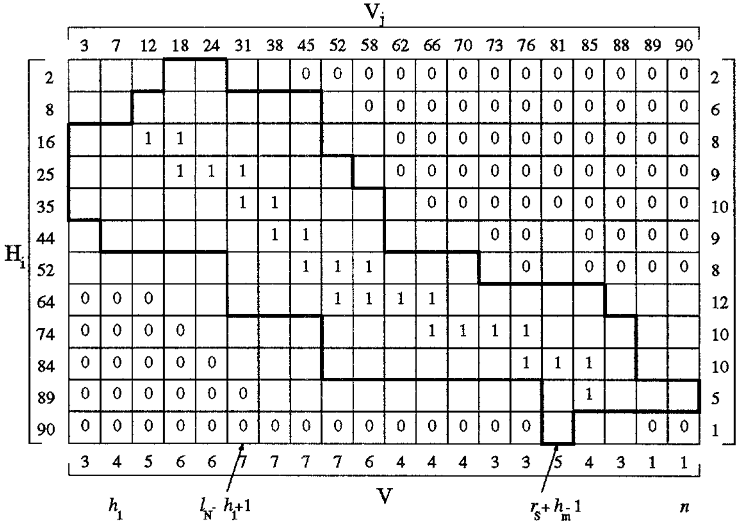

Fig. 7. Kernel expansion and shell reduction by means of operations (2), (3) and (5)

ther to the left of P_N or belongs to it. Therefore, the ordinate d_j is such that $d_j \leq v_{l_N-h_1+1}$ and we obtain the shell reduction:
a.3) $c(i,j) \notin \Lambda$, for $i > v_{l_N-h_1+1}$ and $j \in [1..h_1]$.
We now deal with the columns that contain the foot P_S (i.e., j-th columns, with $r_S + h_m \leq j \leq n$) as we did for the columns that contain the foot P_N. We have the lowest bound of u_j (i.e., $u_j \leq U_j^{(2)}$) and the upmost bound of d_j (i.e., $d_j \geq \min\{D_j^{(1)}, \bar{D}_j^{(1)}\}$, where $\bar{D}_j^{(1)} = m - v_{r_S+h_m-1} + v_j$) (see fig. 6). Moreover, the position n_2 of P_N is smaller than l_N and so the cells of the first row to the right of l_N do not belong to Λ. Likewise, since s_1 position of P_S is greater than r_S, we have that the cells of the m-th row to the left of r_N do not belong to Λ. Figure 7 illustrates the kernel and shell obtained by performing the partial sum operations on the columns.

By symmetry, case (b) (P_S is contained in the first l_S columns and P_N in the last $n - r_N - 1$ columns), is analogous to the previous one. We perform the partial sum operations (1), (4) and (5) by using $U_j^{(1)}$ and $D_j^{(2)}$ (instead of $U_j^{(2)}$ and $D_j^{(1)}$).

Let us now assume that there is a set M of columns having length m. From Proposition 5, we deduce that if these columns are adjacent ($v_r = v_{r+1} = \ldots =$

$v_l = m$) and V's elements are a unimodal sequence, then M belongs to Λ and we can expand the kernel by setting $\alpha := M$. Otherwise, there is no convex polyomino that satisfies (H, V). Therefore, we verify that M is made up of some adjacent columns and V's elements are a unimodal sequence and sizes h_1 and h_m of P_N and P_S have to be greater than the number $l - r + 1$ of M's columns. We have the three cases described in Proposition 5. In the first one P_N is to the left of P_S or vice versa, and we know the two feet's positions. In both cases, we can expand the kernel by putting P_N and P_S into α. Since we know positions n_1, n_2, s_1 and s_2, we can perform the partial sum operations (1)-(5).
In case 3.6 (b) we know the positions n_1 and n_2 of P_N and so we perform the partial sum operations (1), (2). Foot P_S is made up of h_m columns and so the j-th column, with $r - h_m + 1 \leq j \leq r - 1$, belongs to P_S or it is to the left of P_S. Therefore, $d_j = m$ or $d_j \geq D_j^{(1)}$, and so $D_j^{(1)}$ is the upmost bound of d_j, for $1 \leq j \leq r - 1$. Analogously, $D_j^{(2)}$ is the upmost bound of d_j, for $l + 1 \leq j \leq n$. Consequently, we can perform partial sum operations (3), (4) and (5) on the columns not belonging to M. Case 3.6 (c) is symmetric to the previous one.

We wish to point out that, we have to perform some partial sum operations on the columns twice (except 3.6 (b) and 3.6 (c) cases): the first time we assume that P_N is to the left of P_S; the second time, we assume that P_N is to the right of P_S . We proceed in the same way for the rows and use H instead of V. From condition (3.2) (i.e., two pairs of consecutive feet have a non-empty intersection), we can deduce some limitations of feet P_W's and P_E'positions, in addition to the ones obtained from Propositions 4 and 5. As for the columns, we have to perform the partial sum operations on the rows twice: the first time, we assume that P_W is north of P_E; the second time, we assume that P_W is south of P_E.
Fig. 8 illustrates the kernel and shell obtained by performing the partial sum operations on the columns and rows, where we assume that P_W is north of P_E. These operations produce a kernel that can be considered as a "spine" of the convex polyomino.

5 The reconstruction algorithm

In [1], we defined an algorithm that establishes the existence of a convex polyomino Λ satisfying a pair of assigned vectors (H, V), with $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$. The number of possible positions of the four feet is $(n - h_1 + 1)(n - h_m + 1)(m - v_1 + 1)(m - v_n + 1) \leq n^2 m^2$. For each, the algorithm attempts to construct a convex polyomino Λ satisfying (H, V) by means of a procedure that performs some operations (called *filling operations*) and then links our problem to the 2-SATISFIABILITY problem [2], which can be solved in linear time. The procedure's complexity is less than $O(n^2 m^2)$ and, since we perform it for each of possible feet's positions, we can deduce that the algorithm's complexity is less than $O(n^4 m^4)$. In this section, we define a variant of this algorithm by means of the partial sum operations. We start out by performing the partial sum operations on the columns and rows and then apply the previous construction procedure to the kernel and shell obtained by our partial sum operations. The new algorithm's main steps are the following:

H_i \ V_j	3	7	12	18	24	31	38	45	52	58	62	66	70	73	76	81	85	88	89	90	H
2								0	0	0	0	0	0	0	0	0	0	0	0	0	2
8			**1**							0	0	0	0	0	0	0	0	0	0	0	6
16			1	1							0	0	0	0	0	0	0	0	0	0	8
25				1	1	1					0	0	0	0	0	0	0	0	0	0	9
35						1	1					0	0	0	0	0	0	0	0	0	10
44							1	1						0	0	**0**	0	0	0	0	9
52	**0**	**0**						1	1	1					0	**0**	0	0	0	0	8
64	0	0	0						1	1	1	1									12
74	0	0	0	0	**0**							1	1	1	1						10
84	0	0	0	0	0	**0**	**0**								1	1	1				10
89	0	0	0	0	0	0	**0**	**0**									1				5
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				0	0	1
V	3	4	5	6	6	7	7	7	7	6	4	4	4	3	3	5	4	3	1	1	

Fig. 8. The kernel and shell obtained by performing the partial sum operations on the columns and rows.

1. We check to see if (H, V) verifies the conditions (2.1).
2. We calculate the partial sums H_i, V_j for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
3. We determine the feet's limitations (denoted by l_N, l_S, r_N and r_S for P_N and P_S,).
4. If V's elements satisfy the 3.6 (b) or 3.6 (c) condition of Proposition 5 we perform the partial sum operations on the columns. Otherwise, we perform the following two steps: in the first, P_N is assumed to be to the left of P_S; in the second, P_N to the right of P_S. We perform the partial sum operations on the columns in both cases and we proceed as follows:

4.1 if H's elements satisfy the same conditions 3.6 (b) or 3.6 (c), we perform the partial sum operations on the rows and use the "old" algorithm's construction procedure. If these conditions do not hold, in the first, we assume that P_W is north of P_E, in the second we assume that P_W is south of P_E. We deduce some other limitations of these two feet by also using condition (3.2). For each substep, we perform the partial sum operations on the rows and use the "old" construction procedure.

Therefore, the feet have a maximum of four combinations and we perform the partial sum operations on the columns and rows for each; then we apply the construction procedure of the "old" algorithm. Performing the partial sums on the columns and rows involves a computational cost of $O(n\,m)$, while the complexity of the construction procedure is less than $O(n^2m^2)$. Consequently, new algorithm's complexity is less than $O(n^2m^2)$.

6 Conclusions

In this paper, we defined a new algorithm that establishes the existence of a convex polyomino Λ satisfying a pair of assigned vectors (H, V). The algorithm's first step consists of performing the partial sum operations on the feet's four combinations. For each combination, these operations reduce shell β and expand kernel α (see fig. 8). We obtain a convex polyomino "spine". By performing the filling operations on this "spine" we further reduce β and expand α. For instance, by performing the filling operations on the "spine" illustrated in fig. 8 we obtain $\alpha = \beta = \Lambda$. From the "old" algorithm's results (see [1]), it follows:
- if α and β produced by the filling operations are such that $\alpha \not\subset \beta$, then there is no convex polyomino Λ that satisfies (H, V).
- If we obtain α and β such that $\alpha = \beta$ and α is a convex polyomino, then $\alpha = \Lambda$; that is, there is at least one convex polyomino that satisfies (H, V) and we have reconstructed one of them.
- If we obtain α and β such that α and β are two convex polyominoes, with $\alpha \subset \beta$, and the length of the j-th column (the i-th row) is equal to, or smaller than, $2v_j$ $(2h_i)$ for all $j \in [1..n]$ $(i \in [1..m])$, then we can refer to the 2-SATISFIABILITY problem, that can be solved in linear time.

In [1] we proved that the "old" algorithm always produces α and β that verify one of the preceeding conditions, whereas we only have some experimental evidence that the new algorithm produces these results. We ran it up to thousands of cases and found that for each of them, the α and β produced by the partial sum and filling operations verify one of the three conditions. Moreover, this algorithm is much faster than the old one and allows us to reduce the feet combinations to be examined down to four (with the "old" algorithm, we have to examine all the positions of the four feet (i.e., $O(n^2m^2)$ positions). We wish to point out that, if polyomino Λ is "oblong", that is the length of its columns and rows are small with the respect to m and n, then the partial sum operations produce a good expansion of the kernel, that is a big "spine" of the polyomino. On the contrary, if there are some columns and rows of Λ having length about equal to m and n, then the partial sum operations determine a small "spine" of the polyomino. But, in this case the filling operations produce a good expansion of the kernel and a good reduction of the shell.

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