Discrete Image Generation

Circle Digitization and Cellular Automata

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Abstract. The article studies the discrete circles with the help of cellular automata. It is shown that if we consider a discrete circle as a succession of horizontal, diagonal and vertical segments which lean on a beam of parabolas, we can describe it by local properties and so, construct it by cellular automata. In fact, two algorithms are examined. The first one localizes the intersections between a given real circle and the grid. The second one constructs a well-known discrete circle: the arithmetical circle.

Keywords: discrete circles, construction, arithmetical circle, cellular automata

1 Introduction

We propose us to study discrete circles with cellular automata. This work has an interest from the point of view of the cellular automata themselves because it consists in knowing if we can introduce isotropy in an isotropic network such as the two-dimensional grid, in which two directions are privileged. And so, in particular, it puts the question of the judicious utilization of the cellular automata in order to simulate physical phenomena which are for the most part isotropic. But, above all, this work shows that we can obtain good approximations of circles and hence, good approximations of the Euclidean distance, if we consider the discrete circle to be composed by pieces of horizontal, diagonal and vertical straight lines which lean on a beam of parabolas. It allows to describe the discrete circle with local properties. As a matter of fact, we explain the working of two automata, the first one localizes the intersections between the real circle and the grid and, the second constructs a well-known discrete circle: the arithmetical circle.

In the first section, we remember some definitions concerning cellular automata and we make a brief paragraph about the digitization of circles and the link with the discrete parabolas denoted by H_k. The second part deals with the construction of the parabolas H_k and $\bar{H}_k$ by cellular automata. In the following section, we show that these parabolas are the basis of a cellular automaton which localizes the intersections between a given real circle and the two-dimensional grid. The last part explains how we can construct the arithmetical circle with a cellular automaton.

2 Definitions

A *two-dimensional cellular automaton*, A, is a 4-tuple $(2, S, H, \delta)$ such that:

- S is a finite set, the elements of which are called the *states*, denoted by: $S = \{s_k;\ k \in \{0, ..., |S| - 1\}\}$,
- H is a finite subset of $\mathbb{Z}^2$, called the *neighborhood* and denoted by: $H = \{v_j = (x_j, y_j);\ j \in \{1, ..., |H|\}\}$,
- δ is a function from $S^{|H|}$ to S, called the *local transition function.*

At each point of $\mathbb{Z}^2$, there is the same finite automaton.
A *configuration* C_A of the cellular automaton A is an application from $\mathbb{Z}^2$ to S.
For all t that belong to $\mathbb{N}$, the configuration C_A^t becomes the configuration C_A^{t+1} defined by:
$\forall (x, y) \in \mathbb{Z}^2,\ C_A^{t+1}(x, y) = \delta(C_A(x + x_1, y + y_1), ..., C_A^t(x + x_{|H|}, y + y|H|))$.
The state of a cell at time $t + 1$ is, locally computed according to its state and the states of its neighbor cells at time t, via δ.
In this paper, we only use is the Moore's neighborhood.

3 Circle digitization

There exist different ways to define the discrete circle. The more natural of them consists in considering the real circle, mapping it to a square grid and finding its digitization. In this section, we first give a formal definition of the digitization of a circle and then, we explain the link between the circle and the parabolas $y = \sqrt{2kx + k^2}$.

Let us formally define the digitization of a real object.
Let S be a finite set of $\mathbf{R}^2$. We can associate to S a finite set of points of $\mathbb{Z}^2$. This set, denoted by $I(S)$ is called the *discrete image* of S, and the projection which transforms S into $I(S)$ is called the *digitization*. If we consider the object to be a circle, we obtain the following definition of the discrete circle.
A discrete circle, centered on $O(0, 0)$ and the radius of which is R, is a set of points of the square grid that is a digitization of the real circle the center of which is O and the radius of which is R.
For example, the figure 1 (a) shows a discrete circle. Indeed, it is the set of integer points which is obtained by the digitization which is called "grid digitization" in [2]. The figure 1 (b) represents the real circle which has been used to obtain the figure (a).
Notice that in the following, we suppose that the radius of the real circle is an integer and the center is the origin of the coordinates system.

In fact, a discrete circle can be seen as a succession of horizontal, diagonal and vertical segments. The figure 1 c) gives an example. In the first octant (it is symmetric in the others), a method which can help to determine the segments is the following one: if we consider the point the first coordinate of which is

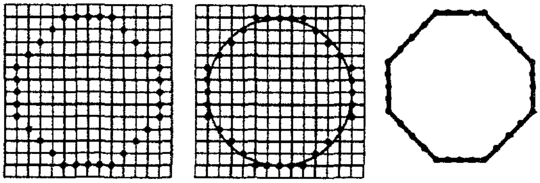


Fig. 1. Grid digitization (a) and (b), a discrete circle is composed by a succession of horizontal, diagonal and vertical straight lines (c).

$x = R - k$ with $k \geq 1$ (k an integer), the equation of the circle implies that the second coordinate is an integer near the value $y = \sqrt{2kx + k^2}$ ([3]). So, according to the choice of the digitization of the circle, we obtain a digitization of the parabola $y = \sqrt{2kx + k^2}$ and, the type of the segments are determined by their position in relation to these discrete parabolas.

4 Parabolas H_k et $\bar{H}_k$

We denote h_k the parabola the equation of which is $y = \sqrt{2kx + k^2}$.
$\mathcal{H}_k = \{(x, y) \in \mathbf{R}^2 / y = \lfloor\sqrt{2kx + k^2}\rfloor\}$ and $\bar{\mathcal{H}}_k = \{(x, y) \in \mathbf{R}^2 / y = \lceil\sqrt{2kx + k^2}\rceil\}$ are two possible digitizations of the parabola h_k. In the following text, we will pay attention to the broken straight lines which respectively join the points of $\mathcal{H}_k \mid^2_{\mathbf{N}}$ and $\bar{\mathcal{H}}_k \mid^2_{\mathbf{N}}$. Notice that we improperly use the term "discrete parabolas" when we mention them. Let us assume that k is an integer greater or equal to 1. This section is devoted to the construction of the parabolas $\bar{H}_k$.

In [4], we have studied the beam of parabolas H_k and we have proved the following lemma.

*Lemma 1. Let C_0 be the initial configuration such that all the cells of the discrete plane are in the quiescent state except the cell $(0,0)$.
There exists a cellular automaton which starting from the configuration C_0, constructs in real time the parabolas H_k for $k \geq 1$.*

We could try to construct the parabolas $\bar{H}_k$ with a similar method to the one which we used for H_k, that is to say to simultaneously construct the parabolas $\bar{H}_k$ and their symmetries with respect to the first diagonal. But, there exists a simpler one. As a matter of fact, it is sufficient to remark that for all x which are not integers, $\lceil x \rceil = \lfloor x \rfloor + 1$ and, for all x integer, $\lceil x \rceil = \lfloor x \rfloor$. These equations infer a method to construct the parabolas $\bar{H}_k$ from the parabolas H_k.
For all x real, we have:

$$\begin{cases} \bar{H}_k(x) = H_k(x) \quad \text{if} \quad h_k(x) \text{ is an integer} & (1) \\ \bar{H}_k(x) = H_k(x) + 1 \text{ else} & (2) \end{cases}$$

where $H_k(x) = \lfloor\sqrt{2kx + k^2}\rfloor$, $\bar{H}_k(x) = \lceil\sqrt{2kx + k^2}\rceil$ et $h_k(x) = \sqrt{2kx + k^2}$.
But, the equation (1) is equivalent to $2kx + k^2 = m^2$ with m an integer. Hence,

if the point (x, y) (x and y integers) belongs to the parabola H_k then the point $(x, y+1)$ belongs to the parabola $\bar{H}_k$ except when $2kx + k^2$ is a perfect square. The goal of the following paragraph is to give characterization of the points (x, y) such that $y^2 = 2kx + k^2$ which is adapted to the cellular automata. Then we describe the cellular automaton that locates these points and the last one is devoted to the construction of the parabolas $\bar{H}_k$.

We prove the following lemma:

Lemma 2. *The points (x, y), such that x and y are integers, which verify the equation: $y^2 = 2kx + k^2$ (3) , for all fixed $k \geq 1$, are the points (x_n, y_n) with $n \geq 0$ such that:*

- *for k even: $(x_n, y_n) = (\frac{k}{2}n^2 + kn, k(n+1))$*
- *for k odd: $(x_n, y_n) = (2kn^2 + 2kn, k(2n+1))$*

Proof. The proof consists in considering the two cases : k odd and k even.

Now, we explain how we can locate the points (x_n, y_n) with a cellular automaton. First we examine the following question: how can we obtain the point (x_{n+1}, y_{n+1}) from the point (x_n, y_n)? Then, we deduce a cellular automaton which locates the points (x_n, y_n) as we suppose the parabolas H_k already constructed. And finally, we obtain the automaton that simultaneously constructs the parabolas and locates the points.

i) Relation between (x_n, y_n) and (x_{n+1}, y_{n+1})
   - <u>k odd</u>
     We remark that the expression of $x_{n+1} - x_n$ can be written:
     $x_{n+1} - x_n = 2(k(2n+1)) + k(2(n+1)+1) - k(2n+1)$
     This means that $x_{n+1} = x_n + 2y_n + (y_{n+1} - y_n)$.
     Let D_1 be the straight line defined by the point (x_n, y_n) and the slope -1. The intersection of D_1 with the axis of the first coordinates is the point $(x_n + y_n, 0)$. Let D_2 be the straight line defined by $(x_n + y_n, 0)$ and the slope 1. This straight line contains the points the coordinates of which are $(x_n + 2y_n, y_n)$ and $(x_n + 2y_n + (y_{n+1} - y_n), y_{n+1}) = (x_{n+1}, y_{n+1})$. The figure 2 (a) gives a graphical representation of the relation between (x_n, y_n) and (x_{n+1}, y_{n+1}).
   - <u>k even</u>
     As previously, we remark that the expression of $x_{n+1} - x_n$ can be written:
     $x_{n+1} - x_n = kn + \frac{3k}{2}$.
     This is equivalent to $x_{n+1} = x_n + 2(\frac{y_n}{2}) + \frac{y_{n+1} - y_n}{2}$
     Let D_1' be the straight line defined by the point (x_n, y_n) and the slope -2. D_1' intersects the axis of the first coordinates at point $(x_n + \frac{y_n}{2}, 0)$. Let D_2' be the straight line defined by $(x_n + \frac{y_n}{2}, 0)$ and the slope 2. D_2' contains the points $(x_n + 2\frac{y_n}{2}, y_n)$ and $(x_n + 2\frac{y_n}{2} + \frac{y_{n+1} - y_n}{2}, y_{n+1}) = (x_{n+1}, y_{n+1})$. Let see the figure 2 (b).

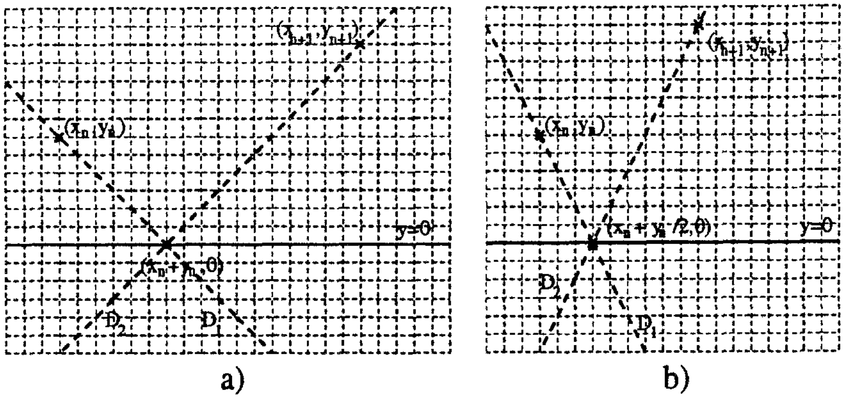


Fig. 2. Relation between (x_n, y_n) and (x_{n+1}, y_{n+1}), k odd (a) and k even (b).

ii) Cellular automaton that locates the points (x, y) such that $y^2 = 2kx + k^2$, the parabolas H_k are supposed to be known.

The straight lines D_1, D_2, D'_1 and D'_2 we have described in the previous part, will correspond to the traces of the signals of the cellular automaton. We suppose that the parabola H_k has already been constructed by the method described in [4]. First, we notice that the point $(0, k)$ verifies the equation (3). So, this point will be the starting point of one of the two signals that we are going to describe now.

Let $S_{k\ even}$ et $S_{k\ odd}$ be two signals respectively composed by two parts:

- a descendant phase; the slope of $S_{k\ odd}$ is -1 and the slope of $S_{k\ even}$ is -2
- an ascendant phase; the slope of $S_{k\ odd}$ is 1 and the slope of $S_{k\ even}$ is 2

The ascendant phase begins when the signal reaches the axe of the first coordinate.

But, since the part 4, if $S_{k\ even}$ (respectively $S_{k\ odd}$) is initialized by a point (x, y) such that $y^2 = 2kx + k^2$ then, the meeting point between $S_{k\ even}$ (respectively $S_{k\ odd}$) and the signal $S_{\lfloor\sqrt{2kx+k^2}\rfloor}$ is a point that verify (3). The figure 3 gives a representation of the signals S_1 and S_2.

So, we have the following lemma:

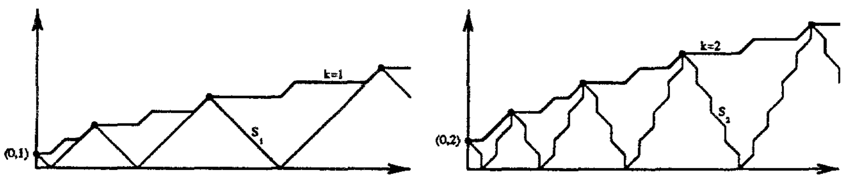


Fig. 3. S_1 et S_2.

Lemma 3. There exists a two-dimensional cellular automaton that locates for all $k \geq 1$, all the points (x, y) such that $y^2 = 2kx + k^2$, if we suppose the parabolas H_k to be given.

Notice that in the following of the text, we use the notation S_k which means $S_{k\ odd}$ or $S_{k\ even}$.

iii) Cellular automaton that simultaneously constructs the parabolas H_k and locates the points (x_n, y_n).

We consider the cellular automaton that constructs the parabolas H_k for $k \geq 1$. We suppose that we add to this automaton the generation of a signal S_k by the point $(0, k)$ as soon as it is constructed, and its propagation. Then, the points (x, y) such that $y^2 = 2kx + k^2$ can be located. Notice that the signals S_k do not mix each others. Indeed, the signals S_k and $S_{\lfloor\sqrt{2kx+k^2}\rfloor}$ are initialized at the same time by the cell $(0, k)$ and they are constructed with the same speed. So, if a signal S_l $(l > k)$ meets a signal $S_{\lfloor\sqrt{2kx+k^2}\rfloor}$, this last one will already be constructed at the meeting point and the signal S_l will continue its ascendant phase.

Then, we have the following lemma:

Lemma 4. Let C_0 be the initial configuration such that all the cells of the discrete plane are in the quiescent state except one: the cell $(0, 0)$.
There exists a two-dimensional cellular automaton that constructs the parabolas H_k for $k \geq 1$ and locates the points (x, y) such that $y^2 = 2kx + k^2$ from the configuration C_0.

In order to obtain the cellular automaton which constructs the parabolas $\bar{H}_k$ from the parabolas H_k, it is sufficient to slightly modify the previous automaton in order to generate not only the parabolas H_k for $k \geq 1$, but also the parabolas $\bar{H}_k$. Indeed, we have seen that the parabola $y = \lceil\sqrt{2kx + k^2}\rceil$ is always one cell above the parabola $y = \lfloor\sqrt{2kx + k^2}\rfloor$ except at the points (x, y) such that $y^2 = 2kx + k^2$. So, a signal denoted by $S_{\lceil\sqrt{2kx+k^2}\rceil}$, which is initialized by the cell $(0, k)$, which propagates with a delay of one unit of time one cell above the signal $S_{\lfloor\sqrt{2kx+k^2}\rfloor}$ and which meets this one at the points (x, y) such that $y^2 = 2kx + k^2$, exactly corresponds to the parabola $y = \lceil\sqrt{2kx + k^2}\rceil$.

Lemma 5. Let C_0 be the initial configuration such that all the cells of the discrete plane are in the quiescent state except the cell $(0, 0)$.
There exists a cellular automaton which constructs the parabolas H_k and $\bar{H}_k$ for $k \geq 1$ from the configuration C_0.

In the two parts which follow, we will see that the discrete parabolas H_k and $\bar{H}_k$ are the basis of the study of the discrete circles with cellular automata.

5 Cellular automaton which determines the intersection between the real circle and the grid

In this part, we show that the previous study concerning the parabolas H_k and $V_{k'}$, allows to construct a cellular automaton which solves the following problem:

given a real circle with a fixed radius ($r \geq 0$), we want to know, for each square of the grid intersected by this circle, how it is crossed.
Let us give a name at each vertex and each edge of a square of the grid. We denote by V_0 the uppermost right vertex and we suppose to go around the square in the trigonometric direction. The figure 4 a) gives a representation of such a square. Then, our goal is to define a cellular automaton which determines the vertices and the edges that are crossed by a given real circle.

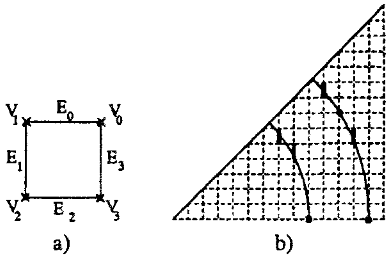


Fig. 4. A square of the grid (a), intersections with the grid for fixed first coordinates (b).

5.1 Intersections between the real circle and the grid

In the following text, we just study the intersections between the first octant of the circle and the grid; the other octants can be obtained by symmetry. So, we are interested in the squares the vertices of which have integer coordinates (x, y) such that $x \geq 0$ and $y \leq x + 1$. In this part, we study the different possibilities for a real circle to cross a square of the grid. First, we can make a simple remark: for a fixed first coordinate x, either the circle crosses a vertex or, it crosses a vertical edge of a square. The figure 4 b) shows these intersections.

Let V be a vertex of a square S. We denote by (x_V, y_V) its coordinates. Let C be the real circle. We assume its radius to be equal to r. We suppose that the point (x_V, y_V) belongs to the circle C. So, we have the following equality: $x_V^2 + y_V^2 = r^2 \quad (4)$ where x_V and y_V integers.
But, (4) $\Leftrightarrow y_V^2 = 2kx_V + k^2$ with $k = r - x_V$. So, at the first coordinate x_V, that the circle meets the point V is equivalent to $y_V = \sqrt{2kx_V + k^2}$. This kind of points have already been studied in the section 4; we have constructed a cellular automaton which locates them. Consequently there exists a cellular automaton that locates the sites where the circle meets a point of the grid.

We denote (α, β) the point which is the intersection between the circle and the line $x = \alpha$. We suppose the point (α, β) is such that β is not an integer. We denote by S the square of the grid such that (α, β) belongs to the edge $[V_2, V_3]$ of this square. We have $\alpha^2+\beta^2 = r^2$ (5) with α integer but β not integer. And, (5) $\Leftrightarrow \beta^2 = 2k\alpha + k^2$ with $k = r - \alpha$. As the points V_2 and V_3 are the extremities of the edge, they are such that: $y_{V_2} = \lceil\sqrt{2k\alpha+k^2}\rceil$ and $y_{V_1} = \lfloor\sqrt{2k\alpha+k^2}\rfloor$. But in the previous section, we have seen that there exists a cellular automaton which construct the parabolas the equation of which are $y = \lfloor\sqrt{2kx+k^2}\rfloor$ (denoted by H_k) and $y = \lceil\sqrt{2kx+k^2}\rceil$ (denoted by $\bar{H}_k$). Consequently, we can also localize with a cellular automaton the places where the circle crosses an edge of the grid.

5.2 Cellular automaton which localizes the circle

Let $C(O,r)$ the real circle centered on $O(0,0)$ and the radius of which is r. The goal of this subsection is to prove the following lemma.

Lemma 6. *There exists a two-dimensional cellular automaton which indicates the intersections between a given real circle and the square grid, the initial configuration is such that all the cells are in the quiescent state except the cells* $0(0,0)$ *and* $(r,0)$.

Proof. The final configuration we want is, for example for circle of radius 13, the one which is shown in the figure 5 (a). An edge is crossed by the circle if

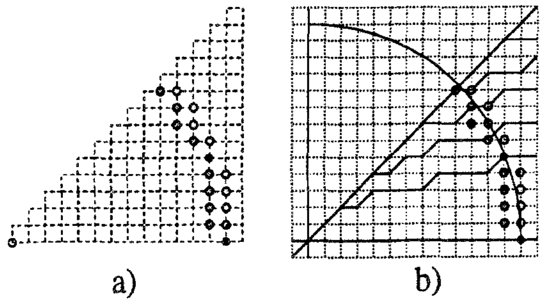


Fig. 5. Final configuration for the circle of radius 13 (a) and the real circle which corresponds (b).

and only if its two vertices are not quiescent (this is indicated by a point on the figure). The black points correspond to the vertices of the squares that are intersected by the real circle. The figure 5 (b) shows where the real circle the radius of which is 13, is exactly localized.

Let us study how the two signals, the one which is composed by the points of color light grey and the other composed by the points dark grey (the black points are the intersection of these two signals), evolve. First, these signals are both initialized by the cell $(r,0)$. If at the first coordinate x $(x < r)$ the real circle cuts a vertical edge, we have seen in 5.1, that the vertices v_2 and v_3 of the intersected square belong respectively to the parabolas $\bar{H}_k$ and H_k with $k = r - x$ and, if the circle meets a vertex denoted by (x,y), this one is such that $H_k(x) = \bar{H}_k(x)$.

It is the same thing for the first coordinates $(x-1)$. Notice that between the first coordinates x and $(x-1)$, the circle only intersects horizontal edges.

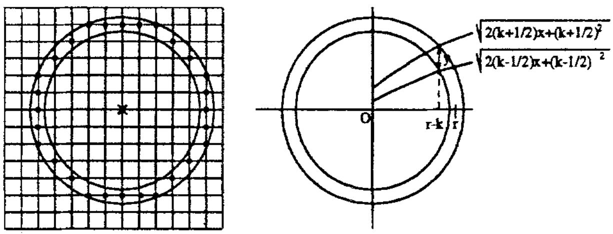
6 The arithmetical circle

In this part, we show how we can construct a well-known discrete circle: the arithmetical circle. First, we give its definition. In a second part, we show the link with the study of the parabolas done in the section 4. Hence, we show how to construct the arithmetical circle with a cellular automaton.

The definition of such a circle has been given by Andres in [1] : the arithmetical circle $\mathcal{C}(0,R)$ centered on O and the radius of which is the integer R, is the solution of the following diophantian equations: $(x,y) \in \mathcal{C}(O,R) \Leftrightarrow x^2+y^2 \in [(R-\frac{1}{2})^2,(R+\frac{1}{2})^2[$, where $x,y \in \mathbb{Z}$ and $R \in \mathbb{N}$

The points of the arithmetical circle are the integer points which belong to the ring the radius of which is R and the width of which is 1. The figure 6 a) gives the example of the circle of radius 5.

Notice that the construction of this circle with a cellular automaton implies to



a) b)

Fig. 6. The arithmetical circle the radius of which is 5.

transform its definition into local properties; this transformation is not obvious a priori. The following highlights these properties.

6.1 Link with the parabolas H_k and $\bar{H}_k$

Lemma 7. For all points $(x,y) \in \mathbb{R}^+$ such that $x \geq y$, all integer $k \geq 1$ we have: $(r-\frac{1}{2})^2 \leq x^2+y^2 < (r+\frac{1}{2})^2 \Leftrightarrow 2(k-\frac{1}{2})x+(k-\frac{1}{2})^2 \leq y^2 < 2(k+\frac{1}{2})x+(k+\frac{1}{2})^2$

Proof. We put $x = r-k$. As it consists in simple computations, we do not give them.

The figure 6 b) gives an illustration of this lemma. So, for a fixed first coordinate, we are interested in the integer points which belong to the small vertical segment that is between the real parabolas of equation $y = \sqrt{2(k-\frac{1}{2})x + (k-\frac{1}{2})^2}$ and $y = \sqrt{2(k+\frac{1}{2})x + (k+\frac{1}{2})^2}$. As an integer point which belongs to the real circle of equation $x^2 + y^2 = (R + \frac{1}{2})^2$ does not belong to the arithmetical circle the radius of which is R, we deduce the following lemma:

Lemma 8. For all $(x, y) \in \mathbb{N}$ *such that* $x \geq y$, *all integer* $k \geq 1$ *we have:*
$(r - \frac{1}{2})^2 \leq x^2 + y^2 < (r + \frac{1}{2})^2 \Leftrightarrow$

$$\begin{cases} \bar{H}_{k-\frac{1}{2}}(x) \leq y^2 \leq H_{k+\frac{1}{2}}(x) \textit{ if } h_{k+\frac{1}{2}}(x) \textit{ not integer} \\ \bar{H}_{k-\frac{1}{2}}(x) \leq y^2 < H_{k+\frac{1}{2}}(x) \qquad \textit{else} \end{cases}$$

6.2 Construction of the arithmetical circle with a cellular automaton

In the first part, we show that the parabolas $H_{k-\frac{1}{2}}$ and $\bar{H}_{k-\frac{1}{2}}$ with $k \geq 1$, can be obtained from the parabolas H_α and $\bar{H}_\alpha$ with $\alpha = 2k - 1$ if we group the cells four by four. Hence, to construct the arithmetical circle it is sufficient to consider the integer points which are between these parabolas.

We put $\alpha = 2k - 1$. As the first part of this subsection is valid for $H_{k-\frac{1}{2}}$ and $\bar{H}_{k-\frac{1}{2}}$, we use the notation $I(x)$ which means $\lfloor x \rfloor$ or $\lceil x \rceil$.
Then we have: $y = I(\sqrt{2(k-\frac{1}{2})x + (k-\frac{1}{2})^2}) \Leftrightarrow y = I(\sqrt{\alpha x + \frac{\alpha^2}{4}})$.
If we put $X = 2x$, we obtain

$$y = I(\sqrt{2(k-\tfrac{1}{2})x + (k-\tfrac{1}{2})^2}) \Leftrightarrow y = I(\tfrac{1}{2}\sqrt{2\alpha X + \alpha^2}) \quad (6)$$

But, for all real positive numbers x ($x = \lfloor x \rfloor + \{x\}$ where $\{x\} \in [0, 1[$), we have:

$$\lfloor \frac{1}{2}x \rfloor = \begin{cases} \frac{\lfloor x \rfloor}{2} & \text{if } \lfloor x \rfloor \text{ even} \\ \frac{\lfloor x \rfloor - 1}{2} & \text{else} \end{cases}$$

Indeed, $\lfloor \frac{1}{2}\{x\} \rfloor = 0$ and then, $\lfloor \frac{1}{2}x \rfloor = \lfloor \frac{1}{2}\lfloor x \rfloor \rfloor$.
Likewise,

$$\lceil \frac{1}{2}x \rceil = \begin{cases} \frac{\lceil x \rceil}{2} & \text{if } \lceil x \rceil \text{ odd} \\ \frac{\lceil x \rceil + 1}{2} & \text{else} \end{cases}$$

because $\lceil \frac{1}{2} \rceil = \lceil \frac{1}{2}\lceil x \rceil + \frac{1}{2}(\{x\} - 1) \rceil$ et $\lceil \frac{1}{2}(\{x\} - 1) \rceil = 0$.
If we apply this to (6), we obtain:

$$y = \lfloor \sqrt{2(k - \frac{1}{2})x + (k - \frac{1}{2})^2} \rfloor = \begin{cases} Y = \lfloor \sqrt{2\alpha X + \alpha^2} \rfloor \text{ with X=2x, Y=2y} \\ \text{ if } \lfloor \sqrt{2\alpha X + \alpha^2} \rfloor \text{ odd} \\ Y = \lfloor \sqrt{2\alpha X + \alpha^2} \rfloor \text{ with X=2x, Y=2y+1} \\ \text{ else} \end{cases}$$

and,

$$y = \lceil\sqrt{2(k-\frac{1}{2})x+(k-\frac{1}{2})^2}\rceil = \begin{cases} Y = \lceil\sqrt{2\alpha X+\alpha^2}\rceil \text{ with X=2x, Y=2y} \\ \text{if } \lceil\sqrt{2\alpha X+\alpha^2}\rceil \text{ pair} \\ Y = \lceil\sqrt{2\alpha X+\alpha^2}\rceil \text{ with X=2x, Y=2y-1} \\ \text{else} \end{cases}$$

Then the parabolas $H_{k-\frac{1}{2}}$ and $\bar{H}_{k-\frac{1}{2}}$ are respectively obtained from the parabolas H_α and $\bar{H}_\alpha$ with $\alpha = 2k-1$, grouping the cells four by four as it is indicated in the figure 7.

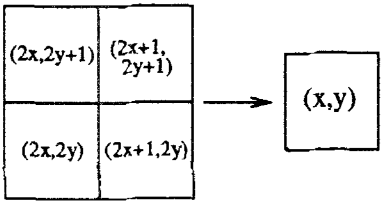


Fig. 7. How to group the cells to obtain $\bar{H}_{k-\frac{1}{2}}$ from $\bar{H}_k$.

Lemma 9. Let C_0 be the initial configuration such that all the cells of the discrete plane are in the quiescent state except the cell $(0,0)$.
There exists a cellular automaton which constructs all the arithmetical circles of radius $r \geq 0$ from C_0.

The figure 8 summarizes the succession of the different constructions which allow to obtain the arithmetical circles. In the first octant, the signal which describes the circle is initialized by the cell $(r,0)$. And, it goes up until it reaches the parabola $H_{\frac{1}{2}}$ except in the case where $h_{\frac{1}{2}}$ is an integer. This is shown by the fact that the parabolas $H_{\frac{1}{2}}$ and $\bar{H}_{\frac{1}{2}}$ meet. Then, it goes on the cell which is up and on the left: it belongs to $\bar{H}_{\frac{1}{2}}$. And so on until it reaches the first diagonal. The figure 8 shows this signal for different sizes of circles.

7 Conclusion

In this paper, we have shown that we can describe some discrete circles with local properties, thanks to the parabolas we have shown up and constructed with cellular automata. We have studied a cellular automaton which localizes the intersections between a given real circle and the grid. Notice that, with this localization, lots of digitizations can be implemented by cellular automata. The second algorithm constructs the arithmetical circle. It shows that we can obtain good approximations of the Euclidean distance using only local properties.
A possible extension of this work is the link with the chamfer distance.

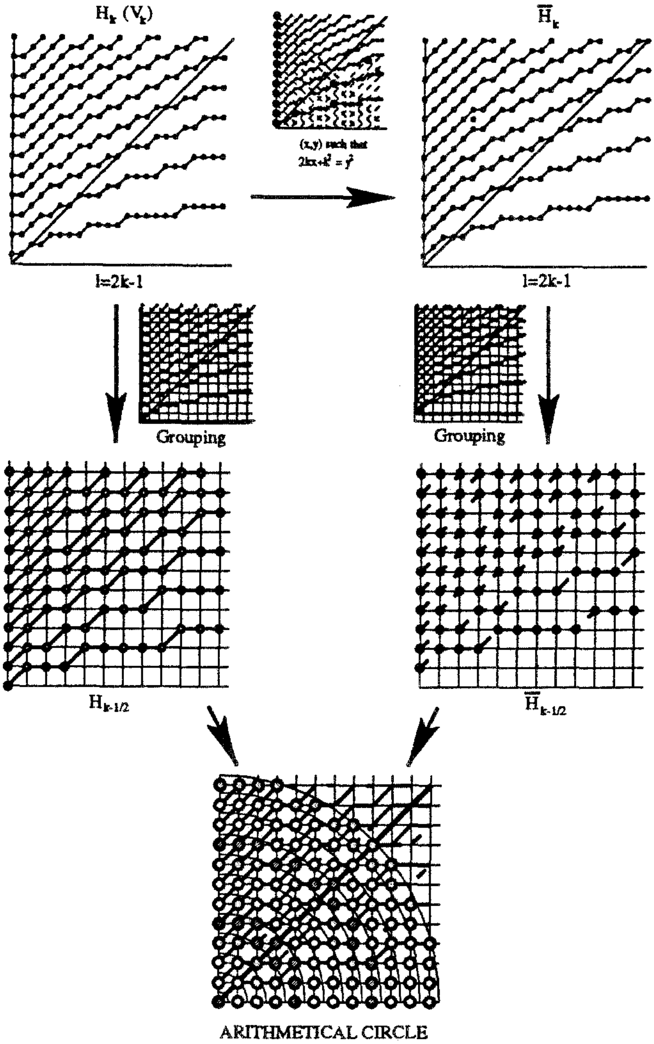


Fig. 8. How can we obtain the arithmetical circle from the parabolas H_k for $k \geq 1$.

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