Discrete Geometry

On Recent Trends in Discrete Geometry in Computer Science

Jean FRANÇON

Laboratoire des Sciences de l'Image, d'Informatique et de Télédétection
Département d'Informatique, 7, rue René-Descartes, F-67000 STRASBOURG
tel. (33)88416343 email: francon@dpt-info.u-strasbg.fr

Abstract. This paper could be called "pieces on recent trends ...". It is a set of remarks and thoughts about the actual and the potential development of discrete geometry, especially in computer imagery, and its interaction with geometrical modelling. My personnal view on some recent trends are exposed. It is rather a critical tutorial that emphasizes what I believe to be significative and promizing trends, among the enormous blossoming of researches and results. Thus, this paper is not a survey. Included remarks are particularly illustrated with papers taken from the DGCI conferences, from 1991 to 1996. General remarks are classified into three groups: On growing 3D discrete geometry; On the convergence of problems and their solutions in the subfields of image analysis and of computer graphics; On the trend toward more discrete geometry. These remarks end with some conclusions for future work. In a second part, the actual and the potential importance of combinatorial topology, arithmetic geometry, discrete linearity and piecewise linearity is studied and discussed. Conclusions, directions, and problems for future work are stated.

1. Introduction

Since 35 years a tremendous development of the fields of computer imagery (CI), picture processing and analysis, picture understanding, pattern recognition, computer vision, computer graphics, image synthesis, computational geometry, takes place [Ros96]. These fields led to the development of discrete (or digital) geometry (DG). The advancement of DG is, nowadays, very varied, even profusing. It recalls the richness of geometry and topology in the mathematics. By DG I mean the set of theories developped in computer science for topological and geometrical questions over a finite or denumerable set of points, without analytical tools; these theories search for strong analogies with euclidean geometry. In order to simplify the reading, I only consider set of points of $\mathbf{Z}^n$, n = 2 or 3. A clear trend exists for extending the results for any n. Other spaces are considered in some works; see [Chas91], [Kong89], [Kong92].

Since the sixties, completely new problems were set down by CI. The following two are archetypal : the existence of a discrete Jordan theorem in the discrete plane for discrete curves, and the recognition of a discrete straight line. These problems are still studied. A superficial historical view could conclude that *nil novi sub sole*. But my

point of view is quite opposite: Nowadays our two archetypal problems, as well as all fundamental problems of discrete geometry in CI, first are posed in new ways, in more adequate mathematical known theories, then are solved with more powerful insight, and last lead to more robust and efficient methods, algorithms, softwares.
It is now known that euclidean topology and geometry cannot be carried into discrete spaces without important distorsions and without a multiplication phenomenon of concepts and theories. Thus, we must carefully distinguish DG and euclidean geometry. The first asserts the foundations of CI. The second is the basis of computer geometry, i.e. geometric modelling and computational geometry. I use the term "computer geometry and imaging" (CG&I) to talk about all these fields together. Note that DG also appeared in theoretical computer science (see [Beau91], [Bers90]), in computer architecture (see [Roz92] and below), and in physics [Riv94].
This paper is an attempt to be pieces of a tutorial that underscores some recent trends that seem to me important and significative for the advancement of DG. Thus, this is not a survey. It is primarily intended for computer scientists working in the area of CI. This paper explains my personnal point of view; this is why it is written in the first person. My remarks want to lead the reader to a more organized, more critical, more coherent insight of a part of DG, particularly of topology. And also to lead to conclusions and directions for future work and to open problems.
Section 2 is devoted to general remaks, concerning global trends, trends transversal to all subfields. Their are threefold: On growing 3D discrete geometry; On the convergence of problems and their solutions in the subfields of image analysis and of computer graphics; On the trend toward more discrete geometry. These remarks are followed by conclusions. They are more precisely illustrated in the sections 3 and 4. In section 3, the actual and the potential importance of combinatorial topology is studied and discussed. Problems and conclusions for future work are stated. Section 4 is devoted to the importance of the discrete linearity and piecewise linearity. General conclusions for future work are stated in section 5.
In the examples and references, I largely flavour the papers of the DGCI conferences, since its beginning in 1991. This conference always wanted to reflect new ideas and trends in DG. This flavour tends to recognize the role played by this conference. In order to avoid a too long list of references, I will give only recent or very significative references; the reader is invited to recurse.

2. General remarks

General, global trends can be organized in three highly interacting groups.

2.1. Discrete 3D and nD discrete geometry

The development of 3D DG is obvious. Its motors are, first medical imaging, second, computer graphics. Medical imaging, and the same 3D imaging in other application domains, cannot be contented by slice by slice processing. New space and time computer ressources allow really 3D methods. In the field of computer graphics, clearly 3D discrete methods are growing: See several DGCI papers on discrete ray tracing and works of Arie Kaufman and others on "voxel machines" (DGCI'96, [Kauf93]).
3D DG is more difficult than 2D, because it is no longer possible to be satisfied by intuitive and approximative reasonnings, as too often in 2D. Thus, 3D DG requires more rigour and mathematical foundations. 3D DG will undoubtly be developped. 4D

DG also, for natural extensions of 3D results to space-time problems. nD also, for the fundamental reason that mathematical extensions from 2D or 3D DG must lead to algorithmic generality, simplicity, robustness, and efficiency; thus, it must lead to better software as well.

2.2. The same DG works in all the computer imagery

Already in 2D, the developments of DG in separated communities of image analysis and image synthesis led to the same concepts and results.

I claim more: I claim a big class of topological problems is essentially the same in CI and in computer geometry; to decribe the topology of a continuous or discrete scene, its construction, modifications, or manipulations, are essentially the same.

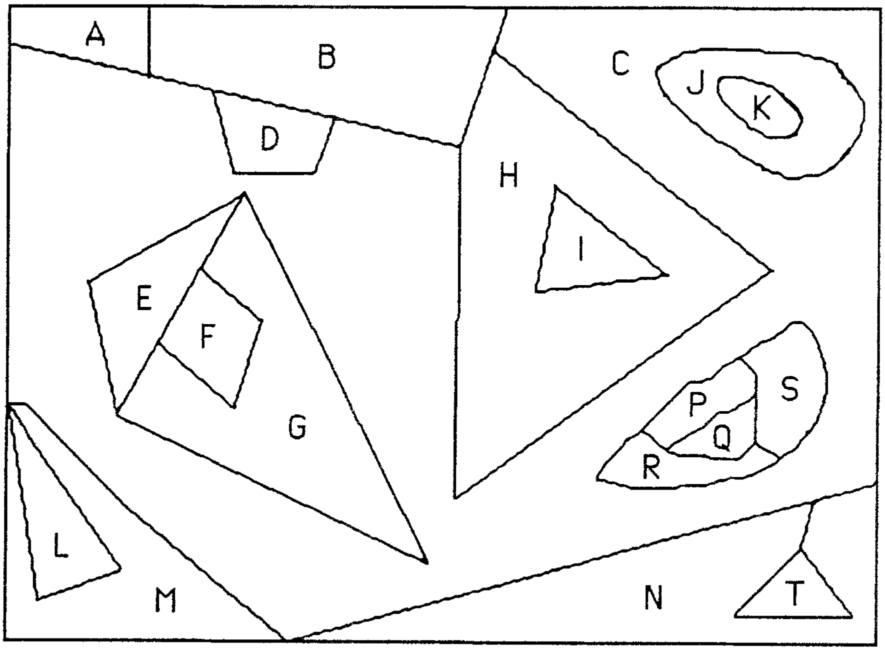

Figure 1. Plane subdivision: cartoon or regions of a picture

The following example, a fundamental one in all CG&I, is used in this paper. A classical problem in image analysis is to compute the adjacency graph of a region segmented picture. A more complicated problem is to take into account multiple adjacencies and possible inclusion of a region in another (enclave). This leads to a structure I call *inclusion tree of the map of the regions* ([Krop95] [Fior95], and DGCI'96 conference). This structure contains more information than a multigraph, because it contains a cyclic ordering of all the edges incident to any vertex. In other terms, it contains all possible adjacence, incidence, and inclusion informations, that are the topological informations of the image considered as a subdivision of the discrete plane.

The same problem is encountered for a while in euclidean geometric modelling (see [Duf89], [Baud89], [Braq91]), e.g. under the name of representation of a *cartoon*. In the field of political cartography, the regions are defined by their boundaries. The problem of describing the topological relations of the regions of a political map is that of describing the topolocal relations of a subdivision of an euclidean plane (or sphere!). It is thus the same problem, details apart.
Let us consider the example of figure 1. Regions are labeled A, B, ..., T; multiple adjacences are visible between C and B (resp. E and G); the sons of the root of the inclusion tree are A, B, C, D, H, L, M, N, T; K is the only son of J; I is the only son of H; C has three sons: J, another with sons E, F, G, and a third with sons P, Q, R, S. A detail is: The common vertex of L and M cannot exist in an image. Another detail is: An isthmus, like the edge linking the boundary of N to that of T, cannot exist in a picture, but can exist in some problems of geometric modelling.
Now, the 3D extension of this problem and of this structure is an open problem: it is difficult and inescapable. In section 3, I will precise the topological concepts used in computer geometry for solving this problem in 2D and 3D.
Conclusion: The convergence of concepts, methods and results comming from works in different fields is a fact. It is to be carried at a higher level, especially for modelling, that can go from discrete to continuous, or continuous to discrete, with very few modification, if any. This point will be developped in section 3.

2.3. From "More Discrete Geometry" to "A Full Discrete Computer Imagery"

2.3.1. In Computer Imagery

Historically, the main DG development comes from image analysis since 1960. Image synthesis seriously contributed to DG only since the late seventies, with the raster graphics capabilities (but note the important exception of curve plotting). In the following I assume computer graphics that use the raster technology. The use of DG in computer graphics can be reduced to rather nothing. Let us take the basic and elementary example of a polygon display, say a triangle. This triangle is given by the coordinates of its vertices in an euclidean plane. Assume the pixels of the display device are the integer points in this coordinates system. The algorithmic problem of the display is then to compute all the integer points inside the triangle (and to set them a color in the image memory). The computation is generally a cartesian geometry computation with real numbers (see any "scan line algorithm" in a computer graphics manual). In this method there is no part of DG; it can be said that the change to the discrete world is performed the latest possible. Other methods discretize earlier, even the earlier possible, beginning by a discretization of the triangle's edges, then by computing the pixels to be displayed by a purely DG algorithm. The general conclusion I want to reach is this one: In computer graphics using the raster technology for display, there is a change to the DG to be made somewhere. This "somewhere" is "adjustable"; thus algorithms can be based on very few DG to DG only.
A general, but diffuse, trend in computer graphics is toward more DG, in the subfields of visualization and modelization. In image analysis, the same trend is obvious.
Ex 1. Voxel machines require DG based algorithms because geometric objects are discrete and stored in the 3D memory.

Ex 2. A font of computer typography is a geometric object represented by its boundary, that is one or several curves, e.g. polygons or splines. In [Pol88] a chip is designed, for a quick display of fonts, that processes discrete curves only, like the discrete triangle above.
Ex 3. Discrete ray tracing is now well known, see DGCI'91 to DGCI'95 and the work of Dany Cohen.
Ex 4. The production of fractal images by the well known Iterated Function System (IFS) method is transformed into a purely discrete method [Krop93]. Other discrete methods for producing fractal images, and more, for defining discrete fractals directly, have been published; see arithmetic geometry below and [Duv95].
Ex.5 In the field of algorithmic geometry, [Chas93] is a very significative example of a directly discrete approach of a difficult problem.
Nevertheless, I do not forget that a lot of discrete problems are solved by *continuation*, i.e. transformation into an euclidean problem, see e.g. DGCI'91 to DGCI'96, [Lore87].
The trend to more DG is also visible in the field of software development, because it is very difficult to make live together distinct methodologies in a software, e.g. morphomath processing and DG processing.

The often searched and finded benefits of the trend to more DG are usually:
1) Time efficiency: basic operations in DG basic algorithms are often more atomic, faster and more repetitive;
2) Algorithms are more simple to design and are more robust;
3) They avoid the frequent dramatic effects (see [Hopc92]) of rounding errors in floating point geometrical computations;
4) The *extension representation* of an object, also called *representation by enumeration*, i.e. by an explicit set of pixels or voxels in a memory, allows to replace intersection computations by detection of an object presence in a given voxel or pixel of the memory. Ex: Euclidean ray tracing requires a huge computation of intersections of rays (euclidean straight lines) and objects (euclidean polygons, spheres,...); discrete ray tracing requires the simulation of a particle movement along a discrete line, and, for each voxel of this trajectory, it requires the detection of an object presence in this voxel. The benefits of this *detection algorithmics* are the three above and others.
One can consider that ex.5 is one of a great class of algorithmic problems, that are particularly difficult in the euclidean geometry but easy in DG. The two following classes are well identified :
1) Squelettization problems;
2) Set (boolean) operations, like union, intersection, difference; their are trivial for extension represented sets.
Thus a wide problem is raised: Is it possible to find a discretization and a continuation such that it could be possible to compose a discretization, a squelettization or a set operation, and a continuation, at least for some application?

2.3.2. The case of arithmetic geometry

The more important and original recent trend in DG is the introduction of the *arithmetic geometry* by Jean-Pierre Reveilles (see the papers by Reveilles and his students in DGCI'91 to DGCI'96). Present arithmetic geometry contains a theory of straight lines and planes (see also section 4), circles, bijective rotations, and discrete affine mappings. The most powerful and promizing theory of discrete linearity is undoubtly that of arithmetic geometry. A surprising aspect of its theory of straight lines and planes is the introduction of a family of these objects parametrized by an integer called the *(arithmetic) width*. We then have practically a possible choice of a working family. Two particular cases play an important role. One is that of *naive* lines, that are 8-connected Jordan curves, and *naive* planes, that are 18-c and 2-manifolds. The other important case is that of *standard* lines, that are 4-c Jordan curves, and *standard* planes, that are 6-c and 2-manifolds (see definition in section 3). More generally, arithmetic geometry, as well as discrete topology, shows a *multiplication effect*: one euclidean concept has several discrete analogs, and there is no reason to favour one; e.g. several bijective rotations have been defined, and all seem to be equally interesting.

Previously, the study of straight lines, circles, planes, ... was done by discretization of the euclidean objects (recurse also from [Kov90], [Kro89]). By this method, results are technically very difficult to obtain, true *tours de force* (see e.g. the result on the structure of discrete straight lines in [Kro89]). In the arithmetic geometry, objects are only defined through integer numbers and integer arithmetic. Real numbers are ignored. Then, reasonnings are purely arithmetical, and often they are elementary. Numerous results of arithmetic geometry are very original. and are unexpected in the discretization approach (see e.g. the non vacuity test for the intersection of two discrete straight lines in [Rev93]). Algorithms are generally simple; they are robust by nature. More, the effect of numerical errors in some floating point computations (e.g. non terminating algorithms by cycling) can be studied and explained through arithmetical geometry (this point deserves more research). Discrete fractals appear naturally. Well identified links exist with theories in physics (see the inflation symetry in quasicristallography, the beautiful paper by Nicolas Rivier [Riv94], and visible discrete straight lines in a daisy).

2.3.3. Conclusion

Arithmetic geometry point out the way for future work. Let me step forward. Why not a full discrete methodology, and a full discrete CI? Why not to avoid the euclidean geometry and the real numbers? Why not working with discrete geometrical objects and discrete operations only. That is, to develop a purely discrete modelling methodology? To develop a discrete geometry intuition and culture? This work requires a sufficiently developped DG with objects and operations analogous to those of the elementary euclidean geometry. This work is, nowadays, not done: but I consider it can be done. In the same spirit, others are developping a computational mechanics which is discrete in time and in mass (but not in space), see [Cad91]. Thus, I consider as a fundamental trend of DG to produce the concept of a full discrete CI.

3. On discrete topology

The two fundamental topological key questions are:

- The study of various concepts of discrete surfaces verifying a Jordan theorem (that is with a connected interior and a connected exterior, such that the interior volume can be defined by its boundary, the surface, for modelling);

- The study of discrete topology equivalence : homeomorphism, homotopy, ... (e.g. for squelettization).

3.1. The two approaches

Two trends are very active. I call the oldest *connectivity based topology* (CBT), because it is based on the classical 2D 4- and 8-connectivity, and 3D 6-, 18-, 26-c. without using concepts of combinatorial topology.

In the 2D CBT, 4-c and 8-c Jordan (also called simple) curves satisfy a Jordan theorem. In the 3D CBT, several notions of a surface have been introduced: [Morg81], [Mala93], [Malg93], discrete polygons in this DGCI'96 conference, and others. In this paper I will use the (26,6)-surfaces of Morgenthaler. A Jordan theorem is proved for the surfaces of Morgenthaler and those of Malgouyres.

In 2D and 3D, a working homotopy theory gives important results for the problem of topology preservation under suppression of one or several points (see the papers by Gilles Bertrand [BerG95] and this DGCI'96 conference).

I call the second trend *discrete combinatorial topology* (DCT), because it is based on the mathematical theory of combinatorial topology [Alex56], with or without the connectivity concepts. In this theory, the combinatorial concept of a surface is that of a 2-dimensional combinatorial manifold, or *2-manifold* for short, *with* or *without boundary*, orientable or not. It will be developped below.

My general remark is: It is time to try to bring closer these two approaches. A first step in this direction is in [Aya95] and is developped below.

3.2. Return to 2D topology

A return to 2D topology is needed because:

- We need to define analogous constructions in a discrete plane and locally on discrete surfaces, e.g. to define the interior of a closed curve on any surface;

- A theory of a discrete plane cannot be only a graph theory because planarity must lie somewhere; more, any theory of a discrete surface must contain a local property of planarity. In my opinion, it must also be a theory of planar graphs or of 2-manifolds (*Problem 0* : Are Morgenthaler surfaces, and other surfaces of the CBT, locally planar graphs in some sense?); 8-connectivity cannot lead to a discrete plane theory because it is not a planar graph.

To make things clear, I take the example of a massively parallel machine, which is a network of processors, each processor being equipped with 4 links [Roz92]; each processor is linked to one to four neighbours, such that the machine is thought of as a graph (maybe the 4-c graph of a window of $\mathbf{Z}^2$). I claim that any reasoning about such a machine uses a reasoning about 2-manifolds. Because in each processor the links are ordered, if no link is free this ordering is sufficient to define an orientable combinatorial map without boundary. That is a representation of a 2-manifold without boundary (the case with free links leads to an orientable 2-manifold with boundary in some cases - a non elementary exercise). Thus, the network of processors can be a topological disk, or an annulus, or a sphere, or a torus, but not a Moebius band, nor a Klein bottle. What are the consequences for programming this network is another question. A partial answer can be found in [Roz92].

In order to step forward I refer to a Jordan theorem in planar graphs given in [Tut84] for a stronger notion of a Jordan curve. Applied to the 4-c graph of $\mathbf{Z}^2$, it gives easily a Jordan theorem for these 4-c curves, with a 4-c interior and a 4-c exterior. Another interesting case is that of a triangulated planar graph. In this case, any Jordan curve satisfies a Jordan theorem, triangles excepted. In this way Jordan theorems can be obtained in a naive and a standard arithmetic plane (see also section 4).

3.3. Combinatorial topology

3.3.1. History

A massive trend of DG is the development of combinatorial topology (CT). The concept of boundary (of combinatorial topology) is in fact used in image analysis since 1970 for 2D imagery, and since 1980 for 3D imagery: A voxel is a cube. Let us call *surfel* a voxel face; the *surfel boundary* of a 3D object (region) is the set of surfels separating a voxel of the object from a voxel of its complementary. Generally, a *surface* is a set of surfels. The theory and use of these surfaces are very actively developped by Herman, Udupa, Rosenfeld, Kong, and others, see [Mig95]. More theoretical advances, mostly on cellular spaces, came recently from several groups, rather independently (in roughly the historical order):

- Vladimir Kovalevsky, in Berlin [Kov90], [Kov92], [Kov93],
- E. Khalimski, R. Kopperman and other mathematicians in New-York [Kha90a], [Kha90b], [Kop91],
- the group of Strasbourg and CIRAD (Montpellier), see DGCI'91 to DGCI'96,
- the group of Montpellier [Ahro95], [Fio95],
- the group of Sevilla and Zaragoza [Aya95],
- and see also [Ken96] in this DGCI'96 conference.

In the same time, theories, data structures and algorithms, softwares, were developped on the basis of CT in the field of *topology based geometric modelling*, and more implicitely in the field of computational geometry. A key concept is *separation of topology and embedding*: A theory must stongly distinguish between a combinatorial structure, e.g. a graph, and its mapping into a space, e.g. $\mathbf{R}^n$ or a compact surface like a sphere or a torus. The fundamental point is here that *a discrete embedding can as well be used.* Thus, the method works for discrete modelling. Intensive use can be made of *combinatorial maps*. These are combinatorial structure well adapted for orientable 2-manifolds without boundary (see 2.2 and 3.2): a *combinatorial map* is a finite set D of objects called *0-cells* (or half-edges, or darts), equipped with two permutations on D; the first one is a pairing operator, for its cycles are of length 2; thus, it defines a set of edges of a graph; each cycle of the second one defines a vertex of that graph together with a circular permutation of the edges incident to this vertex. For 2-manifolds with boundary or non orientable, or 3-manifolds (see below), other combinatorial structures are known, e.g. extensions of combinatorial maps, see [Lien91], [Lien94], [Elt93], [BerY93].

3.3.2. Sketch of the theory

Several ways are possible for introducing CT; let me take informally the one of [Cair68] or [Lef75], restricted to the theory of manifolds (the more general theory is the theory of cellular complexes).

- *Cells*: Let G be a graph; a *0-cell* is a vertex of G; a *1-cell* is an edge of G; a *2-cell* or *face* is a cycle of edges of G (as in a polygon); a *3-cell* or *volume* is a cycle of 2-cells (as in a polyhedron), i.e. each edge of a 3-cell is incident to exactly 2 faces, and each vertex is incident to exactly one umbrella (i.e. a cyclic permutation, up to its sign, of 2 by 2 adjacent faces).

- *Manifolds*: A k-manifold, k=1,2,..., is either a k-cell or is obtained by pairing (k-1)-cells of k-cells:

. By pairing some vertices of 1-cells we get a 1-manifold (combinatorial notion of a set of subdivided Jordan curves), whose boundary is the set of non-paired 0-cells;

. By pairing some edges of 2-cells we get a 2-manifold (combinatorial notion of a set of subdivided surfaces), whose boundary is the set of non-paired 1-cells;

. By pairing some faces of 3-cells we get a 3-manifold (combinatorial notion of a set of subdivided volumes), whose boundary is the set of non-paired 2-cells.

A basic result is: the boundary of a k-manifold is either empty or is a (k-1)-manifold without boundary.

- *Orientability* of a connected 2-manifold is defined by the unicity of obtaining an orientation of any face by propagating a given orientation of a starting given face.

- By taking the 4-c graph of $\mathbf{Z}^2$ and for faces all the minimal cycles (unit squares), we get the 2D cellular space of numerous authors. This space is a 2-manifold without boundary; in this space it is natural to define a region as a connected 2-manifold with boundary; in the same way, a curve is a connected 1-manifold.

- By taking the 6-c graph in $\mathbf{Z}^3$, for faces, taking all the minimal cycles (unit squares), for volumes, taking all the unit cubes, we get the 3D cellular space of [Kov93], [Kha90a], [Ahro95]. This space is a 3-manifold without boundary. It is natural to define in this space a region as a connected 3-manifold with boundary, a surface as a connected 2-manifold and a curve as a connected 1-manifold.

- A classical construction is that of the *barycentric subdivision* of a manifold. Let M be a k-manifold whose vertices are (embedded in) points of $\mathbf{Z}^n$ (or $\mathbf{R}^n$). For any cell of M, compute the barycenter (coordinatewise) of its vertices. By adding these new points, we get a new space, equipped with the incidence graph of cells, and a canonical triangulation of edges, faces, and volumes. This is a new 3-manifold whose volumes are tetrahedra, and faces are triangles. This space is used in [Kop91] for a concept of a discrete surface with a Jordan theorem, and in [Aya95].

3.3.3. More on 2-manifolds

Let us now have a look at 2-manifolds in recent works. 2- and 3-manifolds are constructed in [Ken96]; 2-manifolds are constructed in [Kop91], [Aya95], [Fra95], [Fra96a]. Let me develop two examples.

- The spanish group, in [Aya95], proves that a Morgenthaler (26,6)-surface is a 2-

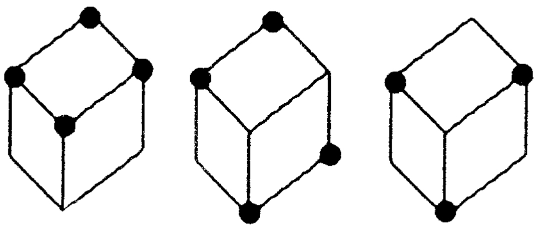

Figure 2. The spanish faces in a unit cube

manifold without boundary. Its faces are the 3 of figure 2, up to the symetries of the cube.
Note that these faces are planar in an euclidean sense. But not any 2-manifold, having only spanish faces, is a (26,6)-surface of Morgenthaler. We are thus led to the
Problem 1. Caracterize the 2-manifolds in $\mathbf{Z}^3$ whose faces are (maybe a subset of) the spanish faces.
- It can be easily proved that a naive plane is a 2-manifold without boundary whose faces are bicubes [Fra96b]. There are 5 bicubes, up to the symetries of the cube, drawn in figure 3 (the drawn cubes are unit cubes).

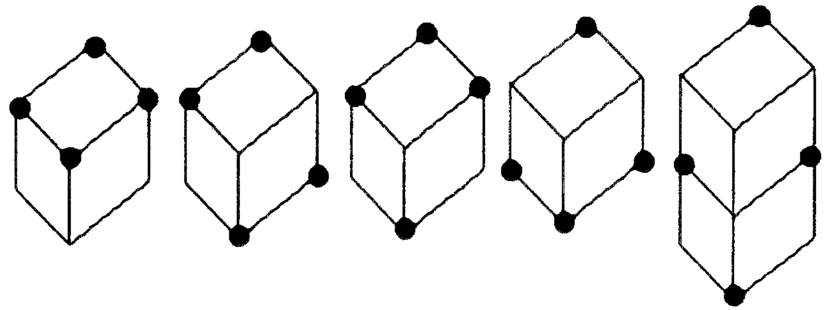

Figure 3. Five bicubes

Note that only 3 are planar in an euclidean sense; but all are planar in a discrete sense.
Problem 2. Caracterize the 2-manifolds in $\mathbf{Z}^3$ whose faces are (maybe a subset of) the bicubes.

3.4. Advantages of the combinatorial topology

The major advantages of CT in DG are:
1) It can be used in different ways (see above).
2) It benefits from over 100 years of intensive mathematical research, in particular:
- The invariant theory of compact 2-manifolds, their correspondance with combinatorial 2-manifolds, their classification; in the case of orientable 2-manifolds the two fundamental results are:
(i) Two 2-manifolds of same genus (number of holes) are topologically equivalent,
(ii) The genus g of a 2-manifold whose number of k-cells is c_k, $k = 0, 1, 2$, is given by the Euler formula

$$c_0 - c_1 + c_2 = 2 - 2g.$$

- Continuous analogs (piecewise linear or convex) and general Jordan theorems are known (see [Kop91], [Ahro95], [Aya95]).
3) Surfaces (2-manifolds) with boundary are easily obtained; this is not the case of the surfaces defined in the framework of CBT.
4) It benefits from over 25 years of research in the field of topology based geometric modelling, with an advanced set of theories, data structures, algorithms, softwares, and methodology (see [Bor93] for a first step).

3.5. Problems

We are thus naturally led to problems. Let me list a few.

Problem 3. Let M be a 2-manifold of some family; develop a theory of preservation of the topology of M under the suppression of one (or more) point; for orientable 2-manifolds without boundary only use the preservation of the genus.

Problem 4. Define Morgenthaler (26,6)-sufaces with boundary in the framework of CBT through the 2-manifolds with boundary whose faces are spanish faces.

Problem 5. Search for 2-manifold caracterization of all the surfaces defined in the framework of CBT; in other words, systematically follow the process of the spanish group.

Problem 6. Let be a region segmented 3D image; if no region is included into another it is a 3-manifold with boundary; if not, it can be described by an inclusion tree of 3-manifolds with boundary; use a theory and a data structure of geometrical modelling to compute this structure; in other words. extend in 3D the problem of section 2.2.

4. Linearity and piecewise linearity

It is claimed in the folklore that *any finite set of points in* Z^2 *or* Z^3 *is piecewise linear*, i.e. locally a straight line or a plane. This is true and well known for any 8-connected or 4-connected curve in the usual discrete plane (see [Kov90], [Krop89], [Deb92]), and even for some 3D curves. A subdivision of a curve into a true 1-manifold is obtained. Recognition of a discrete plane, the first step toward a piecewise linearization of surfaces, is also studied (see [Deb94], [Fra96b]). The possibility of piecewise linearization is also true for a surfel boundary because a surfel can be obviously considered as planar in an euclidean sense or in a discrete sense. In section 3.3. the planarity of the spanish faces, thus the local planarity of the Morgenthaler (26,6)-surfaces, is raised up; the discrete planarity of bicubes, thus the local planarity of any 2-manifold whose faces are bicubes, follows the same way.

But an open problem, surely with non unique solutions, is:

Problem 7. Subdivide a discrete surface, in some sense, into segments that are planar in some sense (e.g. segment of a naive plane, or a standard plane, even an euclidean plane). Subdivide a 2-manifold into a 2-manifold whose faces are planar and edges are straight lines (see a first step in [Bor94]).

The point of view of discrete modelling calls for a notion of discrete polygon, and particularly for a discrete polygon that is also a 2-manifold (with one boundary). in some sense. In the usual 4-c discrete plane, let be M the union of a Jordan curve C and of its interior; it is always possible to piecewise linearize C, then to describe M as a 2D discrete polygon, and finally to solve the problem 7 for a 2D image (done in [Kov90]). But an open problem is:

Problem 8. Define a discrete polygon in any discrete plane by a sequence of its vertices; then, add the condition that a polygon must be a 2-manifold with one boundary.

Other problems could be added, e.g. that of discretization of an euclidean polygon into a discrete polygon (of some given family) (see [Coh95] and this DGCI'96 conference). It is obvious that a lot of CI objects can be reduced to piecewise linear objects, and that the theory of piecewise linearity in DG must progress.

5. Conclusion

Present trends in discrete geometry, that I believe the more important and more promizing, have been described. Problems and conclusions for future work have been given.

I believe that future research will confirm a general trend toward:

- An nD discrete geometry consisting of an arithmetic geometry and, maybe, an unified topology theory;
- A discrete geometric modelling, merged with the present topology based geometric modelling, widely using piecewise linearity;
- A fully discrete computer imagery, in analysis and in synthesis, based on this discrete modelling.

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