On the Enumerative Geometry of Aspect Graphs

Sylvain Petitjean

CRIN-CNRS & INRIA Lorraine
Bâtiment LORIA, BP 239
54506 Vandœuvre-les-Nancy cedex, France
email: petitjea@loria.fr

Abstract. Most of the work achieved thus far on aspect graphs has concentrated on the design of algorithms for computing the representation. After reviewing how the space of viewpoints can be partitioned in view-equivalent cells, we work in this paper on a more theoretical level to give enumerative properties of the different entities entering in the construction of aspect graphs of objects bounded by smooth algebraic surfaces. We show how tools from algebraic geometry can be used to compute their projective characters and other numerical invariants.

1 Introduction

Aspect graphs have been the object of very active research in the last few years. Algorithms have been proposed and implemented for constructing the aspect graph of a wide range of objects, going from polyhedra to solids of revolution and to algebraic surfaces.

Recently, researchers have started to approach theoretical questions of a global kind. In [5], it is shown that the number of nodes of the aspect graph of a smooth algebraic surface of degree d is bounded by $O(d^{12})$ for orthographic projection and $O(d^{18})$ for perspective projection, a special case of a more general result obtained for piecewise-smooth surfaces. This result is the consequence of proving that the degrees of the visual event surfaces are bounded by $O(d^6)$. Up to now, the complexity of these surfaces has been approached from an analytic point of view, i.e. by inspection of the defining equations. In this paper, we examine a new and more abstract approach based on classical enumerative geometry and multiple-point theory which has the advantage of giving exact formulas for the degrees, not just bounds.

The rest of this presentation is organized as follows. Elements of singularity theory are introduced in Sect. 2. Enumerative properties of the visual events are investigated in Sect. 3 in connection with an understanding of prescribed contact of lines and planes with a surface. Then, in Sect. 4, we turn our attention to the study of the main organizers of viewpoint space in view-equivalent cells, i.e., the so-called visual event surfaces. We shall compute their projective invariants and the characters of their singular sets. Finally, Section 5 discusses some of the issues raised by our results, before concluding. While the main ideas of our

Lecture Notes in Computer Science, Vol. 800
Jan-Olof Eklundh (Ed.)
Computer Vision – ECCV '94

approach are intuitively presented in this paper, the details of the derivations can be found in [2].

2 Visual Events

The description of the different visual events in the case of smooth surfaces has been discussed in several papers (see [3] for an introduction), so due to space constraints, we will here mainly state the few important results that we shall need in subsequent sections.

2.1 Projection of a Surface Onto a Plane

Combining results of Whitney and Mather, we infer that for "almost all' (*generic*) observation points, the outline of a smooth embedded surface is a curve whose only singularities are a discrete set of ordinary double points, formed by the transversal superposition of the projection of two fold points, and ordinary cusps, formed by the projection of cusp points. Mather's result asserts that the set of viewpoints for which the preceding result does not hold has measure 0.

2.2 Non-Generic Observation Points

If one chooses the observation point in a special way, one may also obtain some non-generic projections of a smooth surface. These singularities are often classified according to their *codimension*. For "almost all" embedded C^∞-surfaces (*projection-generic* surfaces), it is sufficient to detect the degenerate singularities of codimension less than the dimension of the viewspace considered; more degenerate singularities can only be observed for very special surfaces. Since not much can be said about the geometry of bifurcation sets of general smooth algebraic surfaces, we shall only consider projection-generic C^∞-surfaces in what follows.

For perspective projection, the stable views occupy volumes in a three-dimensional space of viewpoints. The boundaries of these volumes are formed by transitional views of codimension 1, 2, and 3, i.e., surfaces, lines and points.

2.3 Classifying Surface Points

There is a fundamental relation between some special sets of points on the surface X in $\mathbb{P}^3$ and the different visual events. We shall here be mainly interested in the 1-codimensional visual events, since they are the principal organizers of the partitioning of viewpoint space in view-equivalent cells.

Local Events. There are 3 local one-codimensional visual events. A *swallowtail* occurs when the view line is an asymptotic tangent at some point on the flecnodal curve of the surface, and these lines form a *scroll*, i.e. a non-developable ruled surface. *Beak-to-beak* and *lip* events occur when the viewline is an asymptotic tangent at a parabolic point, and these directions form a developable surface.

Multilocal Events. The one-codimensional multilocal events are the *tangent crossing*, *cusp crossing* and *triple point* events, and they occur when the line of sight grazes the surface in n different points (n is equal to 2 or 3). They correspond to ruled surfaces (developable for tangent crossings) in $\mathbb{P}^3$ obtained by sweeping the line of sight while maintaining n-point contact with X, these surfaces being tangent to X along a family of n curves.

3 Enumerative Geometry of the Visual Events

Our main purpose in this section is to give enumerative properties of the visual events. We shall first explain the kind of invariants we intend to compute. Then we show how to obtain these invariants from an understanding of the contacts of sets of lines and planes with the surface. We will here be mainly interested in the 1-codimensional events, but once projection-genericity is assumed, our method can also be developed for singularities of codimension greater than 1.

Most algebraic results are formulated over an algebraically closed field, so from now on calculations are understood over $\mathbb{C}$. In this paper, we merely give an intuitive presentation of some of the key ideas involved in our computations, but the interested reader will find a full account in [2].

3.1 Projective Characters of Curves and Surfaces

We shall be mainly interested in the *elementary projective characters* of the visual event curves and surfaces. These characters are not the sole projective invariants of a variety in $\mathbb{P}^3$, but they have a twofold importance [6]. First, a wide group of problems can be solved using their knowledge alone. Second, several birational invariants are expressible in terms of these characters.

A space curve has only two elementary projective characters: its *degree* n, which refers to the number of points in which it is met by a plane in general position, and its *rank* r, the number of its tangents which meet an arbitrary line. The importance of this pair is illustrated by the fact that the genus g of a non-singular space curve is given by $2g - 2 = r - 2n$.

Consider a surface X in $\mathbb{P}^3$ with *ordinary singularities*, i.e. a *nodal* curve D_b, where two different sheets of the surface meet transversally, with t triple points and ν_2 *pinchpoints* (at which the two tangent planes coincide) on D_b. Such a surface has four elementary projective characters: its *degree* μ_0 (number of points in which it is met by a line in general position), its *rank* μ_1 (rank of a hyperplane section of X), its *class* μ_2 (the number of its tangent planes containing an arbitrary line) and finally ν_2. t, the rank of D_b, the *class of immersion* of D_b in X (number of tangent planes to X at points of D_b which pass through an arbitrary point in space) and topological invariants like the Euler characteristic can all be expressed as functions of μ_0, μ_1, μ_2 and ν_2.

3.2 Lines Having Prescribed Contact with a Surface

Some of the visual event curves on the surface (i.e. the cusp crossing, flecnodal and triple point curves) can specifically be described as the locus of the points of contact of particular sets of lines with the surface. For instance, the cusp crossing curve is the locus of points of contact of those lines that are tangent at one place and asymptotic at another place.

How do we go about studying the properties of a set of lines having prescribed contact with a surface? The general idea starts with the construction of the following two sets. Let G be the space parameterizing lines in $\mathbb{P}^3$ (the *Grassmannian of lines*). Let X be a surface as before and F_X be defined by

$$F_X = \{(x, l) \in X \times G \;/\; x \in l \cap X\}.$$

Let $\bar{q} : F_X \to G$ denote the map induced by projection onto the second factor (i.e., a point (x, l) of F_X is projected onto the point l of G). We let also $\mathbf{a} = (a_1, \cdots, a_k)$ be a k-tuple of positive integers such that $a_1 \leq \cdots \leq a_k$ and $\sum a_j = s$. We now define $S_{\mathbf{a}}(\bar{q})$ to be the set of points (x_1, l_1) in F_X such that there are $s - 1$ other points $(x_2, l_2), \cdots, (x_s, l_s)$ of F_X all having the same image under $\bar{q}$ and also such that the first a_1 points lie infinitely near each other, the next a_2 points lie infinitely near each other, and so on. Then a technique (*stationary multiple-point theory*) developed in [1] allows, if surface genericity is assumed, to compute the characters of $S_{\mathbf{a}}(\bar{q})$ in terms of invariants of F_X and G (known in the literature as the *Chern classes* of their *tangent sheaf*).

Suppose for instance we want to study the curve on X giving birth to the triple point visual event. In the above construction, we let $\mathbf{a} = (2, 2, 2)$ and it is easy to realize that projecting the set $S_{(2,2,2)}(\bar{q})$ onto X yields precisely the visual event curve on the surface. Projecting onto G yields some properties of the set of lines giving rise to the triple point event, the most important of which is the degree of the corresponding visual event surface.

3.3 Planes Having Specified Contact with a Surface

The parabolic and tangent crossing events are best described by imposing a condition on a set of planes. The tangent crossing surface is for instance obtained as the set of lines supporting the points of contact of bitangent planes.

The idea is somewhat similar to that exposed above. Let $\check{X}$ be the *dual surface* of X, i.e. the closure of the set of tangent planes to X. The genericity of X implies that $\check{X}$ is 2-dimensional and is therefore a surface in $\check{\mathbb{P}}^3$, the projective space of planes in $\mathbb{P}^3$ [4]. The morphism $\check{\pi} : X \to \check{\mathbb{P}}^3$ which sends a point of X to its tangent plane is called the *dual map*. The tangent crossing curve TC on X then projects onto a nodal curve D_b on $\check{X}$ and the parabolic curve onto an edge of regression. The degree of TC is computed from that of D_b, and the degree of the tangent crossing surface is the rank of D_b.

4 Properties of Ruled Surfaces

To this point, we have shown how to obtain the degrees of the visual event curves and surfaces. We now move to the study of the projective characters of the visual event surfaces, and their singularities, and we show how these characters can be computed using those of the curves on which they are based.

4.1 Properties of Developable Surfaces

A simply infinite algebraic system Δ of planes is called a *developable of planes* [6]. A line of intersection of two consecutive planes of the system is a *generator*. A *focal point* of Δ is a point of intersection of three consecutive planes.

Now, we shall call a *developable surface* the set of generators of a developable of planes. Such a surface possesses two singular curves: a *cuspidal edge* D_c, locus of its focal points, and a nodal curve D_b. Properties of the cuspidal edge can be investigated using the fact that Δ is the *osculating developable* of D_c, i.e. the system formed by its osculating planes. Now suppose D is the developable surface corresponding to the visual event curve C (which we assume to be non-singular) of genus g on the surface X. Since the tangent plane to D is constant along each generator, its rank μ_1 is equal to $\rho(C)$ the class of immersion of C in X. Its class μ_2 can be seen to be 0. Then D_c and C are in 1-to-1 correspondence, which means that they have the same genus. And every generator of D is tangent to D_c once, which means that the rank of D_c is equal to the degree μ_0 of D, yielding (c is the degree of D_c, κ its number of cusps):

$$c = 2\mu_0 - \mu_1 + 2g - 2, \qquad \kappa = 3\mu_0 - 2\mu_1 + 6g - 6. \tag{1}$$

4.2 Properties of Scrolls

Scrolls have only a finite number of focal points. Thus, their only singularities are a nodal curve D_b, a finite number ν_2 of pinchpoints (the focal points) and a finite number t of triple points on this nodal curve. Such surfaces enjoy the property that their class equals their degree ($\mu_2 = \mu_0$). The number of pinchpoints and the rank μ_1 are given by:

$$\nu_2 = 2(\mu_1 - \mu_0), \qquad \mu_1 = \mu_0(\mu_0 - 1) - 2b, \tag{2}$$

and we shall obtain b, the degree of D_b, in a moment.

4.3 Properties of Ruled Surfaces

As can be inferred from the previous sections, ruled surfaces are specified, as far as their elementary projective characters are concerned, by the degree μ_0 and the rank μ_1 alone. Standard theory on ruled surfaces yields that:

$$\begin{cases} b + c = \frac{1}{2}(\mu_0 - 1)(\mu_0 - 2) - g, \\ t + \kappa + \kappa' = \frac{1}{6}(\mu_0 - 4)((\mu_0 - 3)(\mu_0 - 2) - 6g), \end{cases} \tag{3}$$

where κ' is the number of cusps of D_b. In case the surface is a scroll, $c = \kappa = \kappa' = 0$, and (3) gives b and t, while (2) gives ν_2 and μ_1. In case the surface is a developable, c is obtained by (1), so (3) gives b. κ' is the number of transversal intersections of D_b with D_c, and this number can be shown to be $6b + (c - 3\mu_0)(\mu_0 - 4)$. We obtain t from (3) since we know how to compute κ.

5 Degrees of Visual Event Surfaces

Using the techniques we have presented, we can obtain exact formulas for the degrees of the visual event surfaces in terms of the degree d of the original surface. The asymptotic degree bounds are $O(d^5)$ for cusp crossing and tangent crossing and $O(d^6)$ for triple point, which matches results found by Rieger. Our results for the two local event surfaces, i.e. $O(d^3)$, improve those of Rieger ($O(d^4)$). This is beyond the scope of this paper, but these degrees, that we have obtained assuming genericity conditions, can be generalized to the case of arbitrary smooth surfaces through the use of *Hilbert schemes*.

6 Future Research Directions

To conclude, let us say that we would like to derive similar results for a surface with ordinary singularities, and obtain bounds for piecewise-smooth surfaces. Another problem concerns the extension of some enumerative results to the case of orthographic projection, where the space of viewpoints may be represented by a unit sphere. This is in a large part related to the study of two *line congruences*, i.e. the *asymptotic ray congruence*, to which the local event surfaces belong, and the *bitangent ray congruence*, to which the multilocal event surfaces belong.

References

1. S.J. Colley. Enumerating Stationary Multiple-Points. *Advances in Mathematics*, 66(2):149–170, 1987.
2. S. Petitjean. The Complexity and Enumerative Geometry of Aspect Graphs of Smooth Surfaces. In *Proceedings of MEGA'94*, Santander, Spain, 1994. To appear.
3. S. Petitjean, J. Ponce, and D.J. Kriegman. Computing Exact Aspect Graphs of Curved Objects: Algebraic Surfaces. *International Journal of Computer Vision*, 9(3):231–255, 1992.
4. R. Piene. Some Formulas for a Surface in $\mathbb{P}^3$. In A. Dold and B. Eckmann, editors, *Algebraic Geometry*, volume 687 of *Lecture Notes in Mathematics*, pages 196–235. Springer-Verlag, 1977.
5. J.H. Rieger. On the Complexity and Computation of View Graphs of Piecewise-Smooth Algebraic Surfaces. Technical Report FBI-HH-M-228/93, Universität Hamburg, 1993.
6. J.G. Semple and L. Roth. *Introduction to Algebraic Geometry.* Clarendon Press, 1949.