ON CENTRALITY FUNCTIONS OF A GRAPH

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Abstract: For a connected nondirected graph, a centrality function is a real valued function of the vertices defined as a linear combination of the numbers of the vertices classified according to the distance from a given vertex. Some fundamental properties of the centrality functions and the set of central vertices are summarized. Inserting an edge between a center and a vertex, the stability of the set of central vertices are investigated.

For a weakly connected directed graph, we can prove similar theorems with respect to a generalized centrality function based on a new definition of the modified distance from a vertex to another vertex.

1. Introduction

In many practical applications, it is often necessary to find the best location of facilities in networks or graphs. In this context, a real number f(G,v) is associated with every vertex v of the graph G for the criterion of deciding what vertex is best. The criterion of optimality may be taken to be the minimization of the function f(G,v) with respect to v.

One of the most important problems is to determine what kind of functions is suitable for the measure of centrality of vertices in a graph. It is well-known that the transmission number is an example of such functions. In this survey, the centrality function, a generalized form of the transmission number, is defined as a linear combination with real coefficients of the numbers of vertices classified according to the distance from a given vertex in a connected nondirected graph.

As a fundamental theorem, a necessary and sufficient condition for the function to satisfy the centrality axioms is stated in terms of the coefficients.

Inserting an edge between a center and a vertex, the sets of central vertices settled before and after the edge inserting are generally different. Some stability theorems of the sets of central vertices are presented for a connected nondirected graph.

However the situation often arises where a nondirected graph will not be able to meet various requirements and what is then needed is to introduce a centrality function for a directed graph. For a weakly connected directed graph, a modified distance from a vertex to another vertex is defined as a two-dimensional vector of integer components showing the numbers of forward and backward edges contained in the shortest path with respect to a newly defined order relation. It is shown that the major results for a nondirected graph can be extended similarly to a directed graph with respect to a generalized centrality function based on the modified distance.

2. Transmission Number

Let G be a connected nondirected graph with the set of vertices V. A distance d(u,v) between a pair of vertices u and v in G is defined as the minimum number of edges in a path connecting u and v. We now define $c_0(G,v)$ for every vertex v in G as follows :

$$c_0(G,v) = \sum_{w \in V} d(v,w) \tag{1}$$

The number $c_0(G,v)$ is often refered to as the transmission number[1]. A central vertex v_0 for which

$$c_0(G,v_0) = \min_{v \in V} c_0(G,v) \tag{2}$$

is called a median[1] of the graph G.

3. Centrality Function

Let c(G,v) be a real valued function of vertices of G. Then the function is said to be a centrality function if c(G,v) satisfies the following centrality axioms[2].

Centrality Axioms : If there exist no edges between a pair of vertices p and q in a connected nondirected graph G, the insertion of an edge between p and q yields the graph G_{pq} and the difference

$$\Delta_{pq}(v) = c(G,v) - c(G_{pq},v) \tag{3}$$

for any vertex v in G.

Now the function c(G,v) is called a centrality function if and only if

(i) $\Delta_{pq}(p) > 0$ (4)

(ii) $\Delta_{pq}(p) \geqq \Delta_{pq}(v)$ for any v satisfying

$d(v,p) \leqq d(v,q)$ (5)

for any pair of vertices p and q which are not adjacent. (End)

As a generalized form of the transmission number, we deal with a real valued function c(G,v) as follows :

$$c(G,v) = \sum_{k=1}^{\infty} a_k n_k(v) \tag{6}$$

where $n_k(v)$ stands for the number of vertices whose distances from v are k, and a_k's are real constants.

For the function defined by (6), the following theorem can be proved[3].

Theorem 1 : The function c(G,v) defined by (6) is a centrality function for any graph G if and only if a_k's satisfy

(i) $a_1 < a_2 \leqq a_3 \leqq a_4 \leqq \cdots$ (7)

(ii) $2a_k \geqq a_{k-1} + a_{k+1}$, $(k \geqq 2)$ (8)

(End)

As an illustrative example, suppose

$$a_k = k, \quad (k = 1,2,3, \ldots). \tag{9}$$

It is easily shown that

$$\sum_{k=1}^{\infty} k\, n_k(v) = \sum_{w \varepsilon V} d(v,w) = c_0(G,v) \tag{10}$$

and a_k's given by (9) satisfy (7) and (8). Thus we can conclude that the transmission number is a centrality function.

Let c(G,v) defined by (6) be a centrality function for any connected nondirected graph G. A vertex v_0 for which

$$c(G,v_0) = \operatorname*{Min}_{v \varepsilon V} c(G,v) \tag{11}$$

is called a center of G with respect to c(G,v) or shortly a c-center. Let $S_c(G)$ be the set of all the c-centers of G.

4. Stability Theorems

If a c-center p and a vertex q in G are not adjacent, the insertion of an edge between p and q yields the graph G_{pq} with its set of all the c-centers $S_c(G_{pq})$. Then two cases can occur, either

$$\text{Case A : } S_c(G_{pq}) \subseteqq S_c(G) \cup \{q\} \tag{12}$$

or

$$\text{Case B : } S_c(G_{pq}) \not\subseteqq S_c(G) \cup \{q\} \tag{13}$$

for any vertex p in $S_c(G)$ and q in V. A graph for which case B occurs is said to be unstable with respect to c(G,v).

Case A can be classified into two cases,

$$\text{Case A-1 : } S_c(G_{pq}) \subseteqq S_c(G) \text{ and } p \in S_c(G_{pq}) \tag{14}$$

and

$$\text{Case A-2 : } S_c(G_{pq}) \not\subseteqq S_c(G) \text{ or } p \notin S_c(G_{pq}) \tag{15}$$

for any vertex p in $S_c(G)$ and q in V.

A graph G is said to be stable if case A-1 occurs. A quasi-stable graph is a graph for which case A-2 occurs.

We can then prove the following theorem[4].

Theorem 2 : For any centrality function c(G,v) satisfying $a_2 < a_3$, there exist a quasi-stable graph. (End)

A quasi-stable graph with respect to the transmission number is shown in Fig. 1[4].

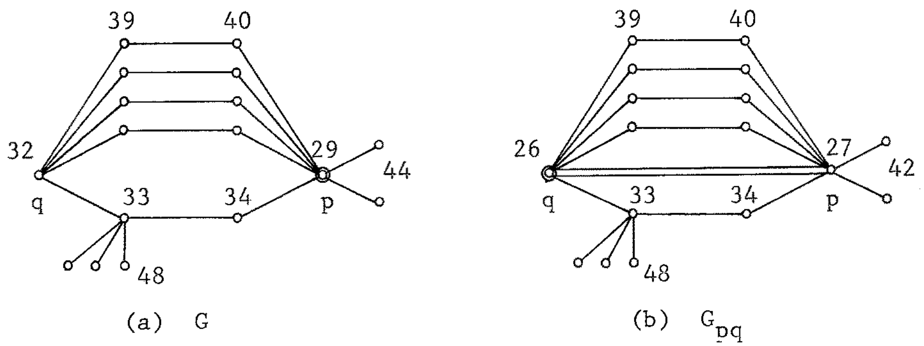

Fig. 1. Quasi-stable graph.

Theorem 3 : For any centrality function c(G,v) satisfying $a_2 = a_3$, all the connected nondirected graphs are stable. (End)

Theorem 4 : Any connected nondirected graph is stable if and only if the centrality function c(G,v) given by (6) satisfies $a_2 = a_3$. (End)

Theorem 5 : For any centrality function c(G,v) satisfying $a_3 < a_4$, there exists an unstable graph. (End)

An unstable graph with respect to the transmission number is shown in Fig. 2[3].

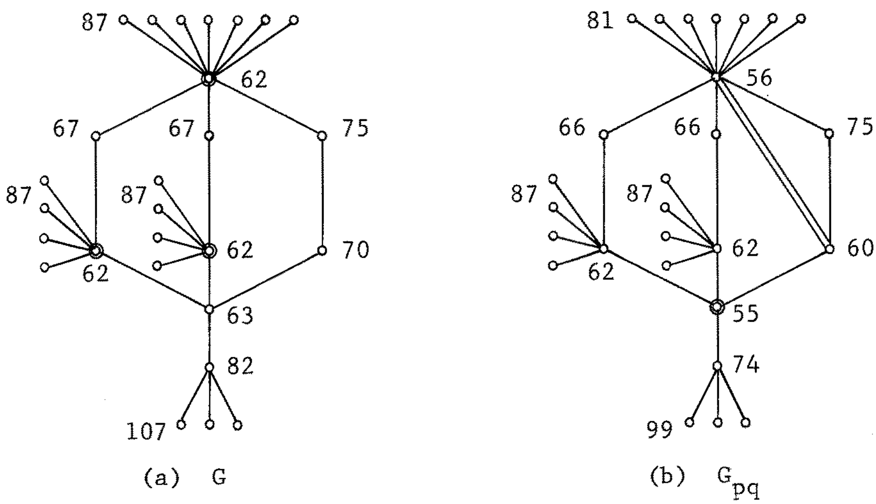

Fig. 2. Unstable graph.

Theorem 6 : For any centrality function satisfying $a_3 = a_4$ all the connected nondirected graphs are quasi-stable or stable. (End)

Theorem 7 : Any connected nondirected graph is not unstable if and only if the centrality function given by (6) satisfies $a_3 = a_4$. (End)

5. Stable Graphs

The theorems in the preceding section show that a centrality function with which all the graphs are stable or quasi-stable is rather trivial one. Characterizing stable or quasi-stable graphs with respect to a given centrality function is an important problem to be solved. The following theorem[2] is basic with respect to the centrality function specified as the transmission number.

Theorem 8 : If a graph G forms a tree, then G is stable with respect to the transmission number. (End)

Let H_k (k = 0,1,2, ...) be the collection of all the connected graphs of nullity k. Then Theorem 8 shows that any graph of H_0 is stable. Since H_2 contains an unstable graph shown in Fig. 2, we may ask if there exists an unstable or a quasi-stable graphs in H_1. Counting the number m of edges in the only loop contained in any graph of H_1, we can define a subset $H_1(m)$ as the collection of graphs containing the single loop of length m.

Recent results with respect to the transmission number include the following two theorems[5].

Theorem 9 : For any $m \leqq 4$, all the graphs of $H_1(m)$ are stable. For any $m \geqq 5$, $H_1(m)$ contains a quasi-stable graph. (End)

Theorem 10 : For m = 7, $H_1(m)$ contains an unstable graph. For $m \leqq 6$, $H_1(m)$ contains no unstable graphs. (End)

The graph shown in Fig. 3 is an example of unstable graph of m=7.

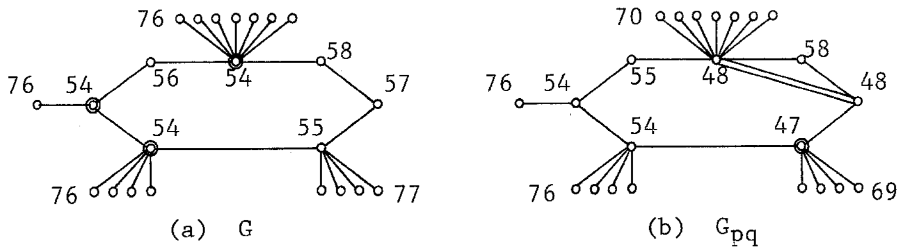

Fig. 3. Unstable graph.

6. Centrality Functions for A Directed Graph

The definitions and the theorems discussed so far can be extended for a directed graph[6]. Let us begin with some preliminary definitions.

Let R^2 be the two dimensional real space defined by

$$R^2 = \{(x,y) \mid x,y \in R\} \tag{16}$$

where R is the set of real numbers. For the simplicity, a vector $(x,y) \in R^2$ is expressed by $x+y\omega \in R^2$, where ω is the symbol specifying the second component.

A natural order and the vector addition can be defined in R^2 as follows.

(i) $x+y\omega > 0$ if and only if $y > 0$ or $y = 0$ and $x > 0$ (17)

(ii) $(x+y\omega)+(x'+y'\omega) = (x+x')+(y+y')\omega$ where $0 = 0 + 0\omega$ (18)

Let N^2 be the subset of R^2 similarly defined with the set of non-negative integer N. It is obvious that R^2 is an ordered abelian group, while N^2 is an ordered semigroup contained in R^2.

Let a directed graph G be weakly connected. A path P between two vertices u and v may be oriented as from u to v. We can then define a vector (a_p, b_p) of integer component associated with the path P where

a_p and b_p are the number of coincide and opposite edges in the path P, respectively. Since (a_p, b_p) can be interpreted as an element $a_p + b_p\omega$ in N^2, we can define a generalized length of the path P such that

$$L_{uv}(P) = a_p + b_p\omega \tag{19}$$

The modified distance from vertex u to vertex v in a weakly connected graph is given by

$$D(u,v) = \min_P L_{uv}(p) \tag{20}$$

where P is an arbitrary path connecting u and v.

Naturally D(u,v) does not fulfil the reflective law, but still satisfies

$$D(u,v) \leqq D(u,w) + D(w,v) \tag{21}$$

Similar to the centrality axioms for a nondirected graph, a centrality function C(G,v) whose values are in R^2 can be defined in terms of the modified distance.

Centrality Axioms : If there exist no edges between a pair of vertices p and q in a weakly connected directed graph G, the insertion of edges from p to q and from q to p yields two graphs G'_{pq} and G''_{pq}, respectively. Let us define

$$\left.\begin{aligned} \Delta'_{pq}(v) &= C(G,v) - C(G'_{pq},v) \\ \Delta''_{pq}(v) &= C(G,v) - C(G''_{pq},v) \end{aligned}\right\} \tag{22}$$

for any vertex v in G.

Now the function C(G,v) is called a centrality function if and only if

(i) $\Delta'_{pq}(p) > 0, \quad \Delta''_{pq}(p) \geqq 0$ (23)

(ii) $\Delta'_{pq}(p) \geqq \Delta'_{pq}(v)$ and $\Delta''_{pq}(p) \geqq \Delta''_{pq}(v)$

for any v satisfying

$$D(v,p) \leqq D(v,p) \tag{24}$$

for any pair of vertices p and q which are not adjacent. (End)

We will deal with the function defined by

$$C(G,v) = \sum_{1<\mu\varepsilon N^2} \alpha_\mu n_\mu(v) \tag{25}$$

where $\alpha_\mu (\varepsilon\ R^2)$ does not depend on G and $n_\mu(v)$ denotes the number of vertices whose modified distance from v are $\mu(\varepsilon\ N^2)$.

Corresponding to Theorem 1, we now obtain the following theorem.

Theorem 11 : The function defined by (25) is a centrality function if α_μ's satisfy

(i) $\alpha_1 < \alpha_2, \quad \alpha_{\mu_1} \leqq \alpha_{\mu_2}$ (26)

(ii) $\alpha_{\mu_2} - \alpha_{\mu_1} \geqq \alpha_{\mu_2+\delta} - \alpha_{\mu_1+\delta}$ (27)

where $1 \leq \mu_1 < \mu_2$ and $1 \leq \delta$. (End)

For a directed graph, we can also prove some stability theorems corresponding to those for a nondirected graph.

7. Conclusion

It has been supposed to be true that any connected nondirected graph is stable with respect to the transmission number [2]. The theorems given here show that the conjecture is false.

Theorem 4 and 6 show that centrality functions with which all the nondirected graphs are stable or quasi-stable are rather trivial. Characterizing stable or quasi-stable graphs with respect to a given centrality function is an interesting problem.

The definitions and theorems of centrality functions for a nondirected graph can be extended for a directed graph, employing the concept of modified distance which seems to be useful in the theory of directed graphs.

References

[1] Christofides, N. : "Graph theory, an algorithmic approach", Academic Press, London, 1975

[2] Sabidussi, G. : "The centrality index of a graph", Theory of graphs, International Symposium, Rome, pp. 369-372, 1966

[3] Kajitani, Y. and Maruyama, T. : "Functional extention of centrality in a graph", Trans. IECE Japan, vol. 59, pp. 531-538, July 1976 (in Japanese)

[4] Kishi, G. and Takeuchi, M. : "On centrality functions of a nondirected graph", Proc. of the 6th Colloq. on Microwave Comm., Budapest, Aug. 1978

[5] Kajitani, Y. : "Centrality of vertices in a graph", Proc. 1979 International Colloq. on Circuits & Systems, Taipei, July 1979

[6] Kishi, G. and Takeuchi, M. : "Centrality functions of directed graphs", Tech. Rep. CST 77-106, Technical Group on Circuit and System Theory, IECE Japan, Dec. 1977 (in Japanese)