TERMINATION TESTS INSIDE λ-CALCULUS

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Abstract

Let be associated to each element N of the set $\mathcal{N}$ of the normal forms of the λ-K-β calculus and to each integer $r > 0$ the semi but non-decidable domain $\mathcal{D}[N,r] \subseteq \mathcal{N}^r$ onto which N, considered as partial mapping $\mathcal{N}^r \to \mathcal{N}$, is total (that is the computation starting from $NX_1 \ldots X_r$ where $N \in \mathcal{N}$ and $X_1,\ldots,X_r \in \mathcal{D}[N,r]$ and evolging through a β-reduction algorithm terminates).The decidability of the relation $\mathcal{D}[N,r] = \mathcal{N}^r$ has been proved in a previous paper. In the present paper, for any N and r, an infinite, decidable subdomain $\mathcal{C}[N,r] \subseteq \mathcal{D}[N,r]$ is defined in a constructive way. The ensuing sufficient condition for the termination of a computation starting from $N X_1 \ldots X_r$ can be tested in a number of steps negligible with respect to those needed for reaching the n.f., if there is one.

1. Introduction

It is well known that λ-K-β calculus (Λ) can be interpreted as a programming language where data, instructions, programs and results are represented by λ-terms [1], [2], [3], [7], [8]. In this way the application of a program to some data is represented by a λ-term (thought as the initial configuration of a computation), whose stepwise reduction, following the β-rule, may represent the associated computation and whose normal form (n.f.), if it exists, shall then represent the result (or the final configuration) of the computation.The set $\mathcal{N} \subset \Lambda$ of n.f.s is a suitable one not only to represent results (univocally determined in consequence of the Church-Rosser theorem [4])but also programs [8] and distinguished data(since any non-α-convertible, closed pair of n.f.s cannot be put convertible without making collapse all closed terms into one [2]). The identification of all programs and data with elements of $\mathcal{N}$ is a first step toward the construction of a model for computation where each program can be written in at most one way. In fact if, for some positive integer r, $N \in \mathcal{N}$, $M \in \mathcal{N}$ and moreover $N X_1 \dots X_r = M X_1 \dots X_r$ for all $X_1, \dots, X_r \in \mathcal{N}^r$ then $N \equiv M$ (extensionality principle [4]). This means that, in such a framework, convertibility may be identified with equivalence of programs with the property that no two programs can be equivalent unless they coincide. The price to pay for that is high:

- the set $\mathcal{N}$ viewed as data set cannot be identified with the set of integers or with some other known data structure set like lists,etc., but it is a larger one
- the set $\mathcal{N}$ viewed as operator or function set contains essentially, even if very sophisticated, only composition operators, with the always present danger to apply some functions to itself, creating paradoxical situations.

Neverthless we find meaningful to study the termination properties of λ -terms of the shape $N X_1 \dots X_r$ where $N, X_1, \dots, X_r$ are n.f.s. In fact this λ-term is the initial configuration of a computation starting from the application of a "program" N to the r-tuple of "data" $X_1, \dots, X_r$. In this case N may also be interpreted as a partial function $N : \mathcal{N}^r \to \mathcal{N}$.

Although the whole domain $\mathcal{D}[N,r]$ onto which $N : \mathcal{N}^r \to \mathcal{N}$ is total is, in general, semidecidable [4] , in [3] it has been shown that, given N and r, the relation $\mathcal{D}[N,r] = \mathcal{N}^r$ is decidable. The theory developed in [3] , neverthless, becomes unseless with respect to termination properties, whenever $\mathcal{D}[N,r] \subsetneq \mathcal{N}^r$. This paper provides, for any pair N,r, a decidable infinite "security" domain $C[N,r]$ (1) $\subseteq \mathcal{D}[N,r]$ onto which $N : \mathcal{N}^r \to \mathcal{N}$ is total.

As a trivial example of the results of [3] we have that, for each free variable a and for all integers r, $\mathcal{D}[a,r] = \mathcal{N}^r$. In the same paper it is defined the whole subset $\mathcal{N}_\omega$ of $\mathcal{N}$ for which this property is true, i.e. such that each n.f. belonging to it behaves, with respect to termination properties, like a free variable.

(1) A first attempt to build $C[N,r]$ is given in [6] . Neverthless the method of [6] is unable to treat some cases.

Moreover we notice that, if N is an arbitrary n.f. and $a_1,\dots,a_r$ are free variables, then surely $Na_1\dots a_r$ possesses n.f. . We can prove (Lemma 1) that $NX_1\dots X_r$ possesses n.f. also when $X_1,\dots,X_r$ are n.f.s which belong to $\mathcal{N}_\omega$. Therefore we can say that, for each n.f. N and integer r, $\mathcal{N}_\omega^r \subseteq C[N,r]$. To find other elements of $C[N,r]$ we may observe that $NX_1\dots X_r$ possesses n.f. also when it reduces to a λ-term in which all redexes are of the shape $MY_1\dots Y_k$ where $k>0$, $M,Y_1,\dots,Y_k$ are n.f.s and $M\in\mathcal{N}_\omega$ or $Y_l \in\mathcal{N}_\omega$ $(1 \le l \le k)$. This last observation suggests us to look for conditions on $N,X_1,\dots,X_r$ which assure us that each bound variable of N will be either non replaced or replaced by n.f.s belonging to $\mathcal{N}_\omega$ in some contractum of $NX_1\dots X_r$. To study the behaviour of the bound variables of a n.f. N we will associate to each bound variable of N a (possibly undefined) list of integers (a.p.)which tells us "where" this variable is bound in N. A further step toward our goal will then be the introduction of the concept of structure, built directly from that one of a.p. taking into account the relative positions of the occurrences of couples of variables in N. The set of all structures of N will finally build the schema of N which may be viewed intuitively as a "map" of the occurrences of the bound variables of N. In this way we will be able to define a necessary condition under which a given variable in an application of n.f.s can be replaced only by a λ-term whose n.f. (if it exists) belongs to $\mathcal{N}_\omega$ (Lemma 2).At this point we will be able to give a definition of $C[N,r]$ (definition [7]) which is based on the comparison between the schema of N and the schemata of all its possible r-tuples of arguments. The formal proof of the correctness of our definition will be given in Theorem 3. We notice that the same definition of schema will be suitable to represent properties of both N and its arguments. This can be justified taking into account the fact that in the contracta of $NX_1\dots X_r$ each X_i $(1 \le i \le r)$ may be, in its turn, applied to other arguments playing so the same role as N in $NX_1\dots X_r$.

2. Key notions and definitions

It is well known that an arbitrary n.f N can be written (unless a finite number of α-reductions) in the following way:

$$N \equiv \lambda x_1 \dots \lambda x_n (\zeta N_1 \dots N_m)$$

where $x_j (1 \leq j \leq n)$, ζ are variables and $N_i (1 \leq i \leq m)$ are n.f.s (2). We call ζ the head variable of N, $N_i (1 \leq i \leq m)$ the i-th component of N and $\lambda x_1 \dots \lambda x_n$ the initial abstractions of N. The order of a variable x in a n.f. N is the maximum number of components of subterms of N whose head variable is x. $|N|$ abbreviates the number of variable occurrences in the n.f. N. Its recursive definition is obviously: $|N| = 1 + \sum_1^m{}_i |N_i|$ when $N \equiv \lambda x_1 \dots \lambda x_n (\zeta N_1 \dots N_m)$. As usual $N[x / X]$ denotes the contractum of $\lambda x N X$. In this case we say that x is replaced by X. According to [1] we say that X is free for x in N iff all free variables which occur in X remain free in $N[x / X]$. Through definition 1 we will associate to each variable which occurs in a n.f. N a list of integers called its access path (a.p.) which tells us, intuitively, what "must be done" to "reach" the variable x and to replace it. The a.p. of x in N is h if x is bound by the h-th initial abstraction of N. This means that to replace x we must apply N to at least h arguments (the h-th argument will then replace x). Let us suppose instead that x is bound by the initial abstractions of the p-th component of a subterm $zY_1 \dots Y_p$ of N, i.e. $Y_p \equiv \lambda y_1 \dots \lambda y_{q-1} \lambda x \bar{Y}_p$, and moreover that the a.p. of z in N is h (i.e., z is bound by the h-th initial abstraction of N). In this case to replace x:

- N must be applied to at least h n.f.s
- if X_h (which will replace z) has at least p initial abstractions, the variable bound by the p-th initial abstraction of X_h must be the head variable of a subterm whose q-th component will finally replace x.

In this case we will define the a.p. of x as the list h,p,q. In the general case, if $Z \equiv yZ_1 \dots Z_p$ is a subterm of N, $Z_p \equiv \lambda y_1 \dots \lambda y_q \bar{Z}_p$ and the a.p. of y in N is μ, then the a.p. of y_q in N is defined as the list μ,p,q.

Lastly we observe that a free variable of N can never be replaced and therefore we assume that its a.p. is undefined. If the a.p. of the head variable of a subterm Z of N is undefined, then the a.p.s of all the va

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(2) For clarity reasons (indexed) x, y, z will denote bound variables, while (indexed) a will denote free variables and variables bound in different abstractions will have different labels. ζ, ∂ and τ will be used for variables either free or bound.

riables bound by the initial abstractions of each component of Z will be undefined too, since all these variables can never be replaced.The formal definition of a.p., then, is the following:

Definition 1. The a.p. (access path) of a variable which occurs in a n.f. $N \equiv \lambda x_1 \ldots \lambda x_n (\gamma N_1 \ldots N_m)$ is a list of integers built up recursively according to the following rules:

i) x_j, for $1 \leq j \leq n$, has a.p. j

ii) the free variables have undefined a.p.

iii) if $Z \equiv \partial Z_1 \ldots Z_p$ is a subterm of N, $Z_p = \lambda y_1 \ldots \lambda y_q \bar{Z}_p$ and the a.p. of ∂ in N is μ (undefined) then the a.p. of y_q in N is μ, p, q (undefined).

Example 1. To replace the variable x_5 in the n.f. $N \equiv \lambda x_1 \lambda x_2(x_2 \lambda x_3(a_1 \lambda x_4(x_3a_2 \lambda x_5(x_5x_4x_4)x_1)))$ we must replace in order:

- the variable x_2 which is bound by the 2-th initial abstraction of N
- the variable x_3 which is bound:
  - in the 1-th component of the subterm $x_2 \lambda x_3(a_1 \lambda x_4(x_3a_2 \lambda x_5(x_5x_4x_4)x_1))$

  and - by the 1-th initial abstraction of this component
- and, lastly, the variable x_5 which is bound:
  - in the 2-th component of the subterm $x_3a_2 \lambda x_5(x_5x_4x_4)$

  and - by the 1-th initial abstraction of this component.

Therefore the a.p. of x_5 in N is 2,1,1,2,1.

In [3] it was introduced the notion of h-replaceability of a variable that, on the ground of definition 1, can now be expressed in the following way:

- a variable is h - replaceable if the first element of its a.p. is lower than or equal to h
- a variable is replaceable if it is h-replaceable for some h > 0
- a variable is non-replaceable if its a.p. is undefined.

Example 2. In the n.f. of example 1 we have that x_5 is 2-replaceable, i.e. it can be replaced only if N is applied to at least two arguments.

Two occurrences of variables in a n.f. N are said to form a <u>couple</u> if they are, respectively, head variable of a subterm Z of N and of one of the components of Z. The notion of couple has been introduced (not explicitly) also in [3]. Moreover in [3] (Theorem 1) it has been shown that if a n.f. N don't contain any couple of variables both h-replaceable, then $\mathcal{V}[N,h] = \mathcal{N}^h$. From this it comes out that the termination properties of applications of n.f.s are strongly dependent on the couples of replaceables variables that occur in them. We will take into account the behaviour of the couples of variables which occur in a n.f. N through the following definition of structure, in which we associate to each couple the a.p.s of its variables.

<u>Definition 2.</u> If $Z \equiv x\, Z_1 \ldots Z_q$ is a subterm of N, μ is the a.p. of x, ν is the a.p. of the head variable y of Z_q then the <u>structure</u> of the couple x, y in N is: $(\mu ; \nu)$.

We notice that different couples can have the same structure.

<u>Example 3</u>. In the n.f. of example 1 we have that the couple of variables x_3, x_5 in the subterm: $x_3 a_2\ \lambda x_5 (x_5 x_4 x_4) x_1$ has structure (2,1,1; 2,1,1,2,1).

Now, if we consider the structures of all couples in a n.f. N we have a complete "map" of the dangerous occurrences of variables in it. We may also limit us to consider the variables which are dangerous when N is applied to exactly r arguments. To this aim we will associate each pair N,r a set of structures (r-schema of N).

<u>Definition 3.</u> Let r be a non-negative integer and $N \equiv \lambda x_1 \ldots \lambda x_n (\zeta N_1 \ldots N_m)$ an arbitrary n.f.

The <u>schema</u> $\mathcal{S}[N]$ of N is the set of the structures of all couples in N (3).

The <u>r-schema</u> $\mathcal{S}[N,r]$ of N is the set of structures so defined:

a) if $r \leq n$ then $\mathcal{S}[N,r] = \{(h,\mu ; k,\nu) \mid (h,\mu ; k,\nu) \in \mathcal{S}[N]$ and $h,k \leq r\}$

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(3) We convent that when different couples have the same structure this one appears only once in $\mathcal{S}[N]$.

b) if $r > n$ and $\mathfrak{z}$ is a free variable then $\mathfrak{S}[N,r] = \mathfrak{S}[N]$

c) if $r > n$ and $\mathfrak{z}$ is a bound variable with a.p. j then $\mathfrak{S}[N,r] = \mathfrak{S}[N] \cup$
$\cup \{(j;\ n+p) \mid 1 \leq p \leq r-n\}$.

We notice that, in case c), the r-schema of N coincides with the r-schema of a n.f. N' such that:

- N' is η-reducible to N
- N' has r initial abstractions.

Example 4. The 1-schema of the n.f. of example 1 is empty. The schema and 2-schema are:

$\mathfrak{S}[N] = \mathfrak{S}[N,2] = \{(2,1,1;\ 2,1,1,2,1);\ (2,1,1;\ 1)\}$.

The 3-schema is:

$\mathfrak{S}[N,3] = \{(2,1,1;\ 2,1,1,2,1);\ (2,1,1;\ 1);\ (2;3)\}$.

etc.

From definition 3 it follows immediately that:

$$\mathfrak{S}[N,0] = \Phi \qquad (4)$$

$$\mathfrak{S}[N,r] \subseteq \mathfrak{S}[N,r+1] \qquad r \geq 0.$$

Moreover if $Z \equiv \partial Z_1 \ldots Z_q$ is a subterm of a n.f. N, μ is the a.p. of ∂ in N and $(\sigma;\nu) \in \mathfrak{S}[Z_q]$ then from definitions 1, 2 and 3 it follows that $(\mu, q, \sigma;\ \mu, q, \nu) \in \mathfrak{S}[N]$.

3. Fundamental properties

In this section we will give properties that generalize some results of [3] and that will be used in the proof of Theorem 3. To render this paper self-contained the classification of n.f.s given in [3] is reported here on the ground of previous definitions.

Definition 4. A n.f. $N \in \mathcal{N}_h$ iff $\mathfrak{S}[N,h] = \Phi$ and $\mathfrak{S}[N,h+1] \neq \Phi$.

Definition 5. A n.f. $N \in \mathcal{N}_\omega$ iff $\mathfrak{S}[N] = \Phi$ and its head variable is free.

(4) Φ denotes the empty set.

Example 5. By definition 4 the n.f. of example 1 belongs to $\mathcal{N}_1$.

The meaning of this classification is that a n.f. N belongs to the class $\mathcal{N}_h$ iff the λ-terms obtained by applying N to h arbitrary n.f.s possess n.f. too, but there exists h+1 n.f.s $X_1,\ldots,X_{h+1}$ such that $NX_1\ldots X_{h+1}$ possesses no n.f. The main theorems of [3] (Theorems 1 and 2) can then be rewritten in the present formalism as follows:

Theorem 1. $N \in \mathcal{N}_h$ $(h \geq 0)$ iff:
- $\mathcal{D}[N,r] = \mathcal{N}^r$ for $0 < r \leq h$
- $\mathcal{D}[N,h+1] \neq \mathcal{N}^{h+1}$.

Theorem 2. $N \in \mathcal{N}_\omega$ iff: $\forall r(r > 0)\mathcal{D}[N,r] = \mathcal{N}^r$.

As sketched informally in the introduction, we show that for each n.f. N the decidable subset $\mathcal{N}_\omega$ of $\mathcal{N}$ belongs to $\mathcal{D}[N,1]$, i.e. we have the following:

Lemma 1 [6] . For each n.f. N $\mathcal{N}_\omega \subset \mathcal{D}[N,1]$.

Proof. We must show that for each $M \in \mathcal{N}_\omega$, NM reduces to a n.f.. If N is λ-free then NM is a n.f.. Otherwise $N \equiv \lambda x_1 \bar{N}$ and $NM \geq \bar{N}[x_1/M]$. In $\bar{N}[x_1/M]$ the subterms which are reducible are possibly only those where M replaces some occurrences of x_1. If we suppose to perform the reductions from the innermost redexes we have that, since $M \in \mathcal{N}_\omega$, these subterms reduce to n.f. . Now we may iterate these considerations on the so obtained λ-term until we have performed all possible reductions in $\bar{N}[x_1/M]$ .Since all subterms of the so obtained λ-term are in n.f., the λ-term itself is in n.f. too. □

Let's now introduce a relation between lists of integers. Through it we will compare schemata and a.p.s of n.f.s.

Definition 6. Two lists of integers $h_1,\ldots,h_p$ and $k_1,\ldots,k_q$ match iff one is an initial segment of the other, i.e.:

$$h_l = k_l \quad (1 \leq l \leq \min(p,q)).$$

Obviously every list matches with the empty list ε .
By extension we will say that a structure (μ ; ν) matches with a list σ if either μ or ν (or both) matches with σ .
Finally Lemma 2 gives a necessary condition on the schema of X_h under which, if $N, X_1, \ldots, X_{h-1}$ are n.f.s, in some contracta of $N\ X_1 \ldots X_h$ a variable having a.p. h, μ in N is replaceable by a n.f. Y such that $Y \notin \mathcal{N}_\omega$.

Lemma 2. Let $N \cong \lambda x_1 \ldots \lambda x_n (\mathfrak{z} N_1 \ldots N_m), X_1, \ldots, X_h$ be h+1 arbitrary n.f.s and x a variable whose a.p. in N is h, μ with $\mu \neq \varepsilon$. Let k be the order of x_h in N. If x in any contractum of $N\ X_1 \ldots X_h$ is replaced by a n.f. Y and $Y \notin \mathcal{N}_\omega$ then $\mathcal{S}[X_h, k]$ must contain a structure matching with μ .

The proof of this Lemma is given in the Appendix.

4. Decision method

Now we are able to give a *sufficient* condition to assure the existence of the n.f. of a λ-term $N\ X_1 \ldots X_r$, where $N, X_1, \ldots, X_r$ are arbitrary n.f.s, without execute any reduction. To this aim we associate to each arbitrary n.f. N and each integer $r > 0$ a domain $\mathcal{C}[N,r]$ whose elements are r-tuples of n.f.s $X_1, \ldots, X_r$.

Definition 7. Let $N \equiv \lambda x_1 \ldots \lambda x_n (\mathfrak{z} N_1 \ \ldots \ N_m)$ be an arbitrary n.f., r a positive integer and t_j $(1 \leq j \leq r)$ the order of x_j in N (5). The r-tuple $X_1, \ldots, X_r \in \mathcal{C}[N,r]$ iff, for each structure $(h, \mu \ ; \ k, \nu) \in \mathcal{S}[N,r]$, at least one of the following conditions hold:

1) if $\mu \neq \varepsilon$ then no structure of $\mathcal{S}[X_h, t_h]$ matches with μ otherwise $X_h \in \mathcal{N}_\omega$
2) if $\nu \neq \varepsilon$ then no structure of $\mathcal{S}[X_k, t_k]$ matches with ν otherwise $X_k \in \mathcal{N}_\omega$.

From this definition it follows that when $\mathcal{S}[N,r] = \Phi$ $\mathcal{D}[N,r] = \mathcal{N}^r$, that is $\mathcal{S}[N,r] = \Phi$ implies that $N \in \mathcal{N}_q$ form some $q \geq r$ (6) by definition 4. In this particular case only, the results of the present method coinci

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(5) Obviously if $n < r$ then $t_j = 0$ for $n < j \leq r$.
(6) ω is considered greatest than any integer.

de with those given in [3] .The following Theorem assures us that the given definition of $\mathcal{C}[N,r]$ is correct, i.e. for any r-tuple $X_1,\dots,X_r \in \mathcal{C}[N,r]$: $N\ X_1 \dots X_r$ reduces to n.f..

Theorem 3. Let N be an arbitrary n.f. and r a positive integer. Then $\mathcal{C}[N,r] \subseteq \mathcal{D}[N,r]$.

Proof. The proof is done by induction on the number q of couples having defined structures in N (called in the sequel d-couples).
First step. q = 0, i.e. $\mathcal{S}[N,r] = \Phi$. In this case $N \in \mathcal{N}_p$ with $p \geq r$. Therefore $N\ X_1 \dots X_r$ possesses n.f. by Theorem 1.
Inductive step. Let us assume that this Theorem is true for $q \leq u$ and we prove that it is true also for q = u+1. I.e.we consider a n.f. N with u+1 d-couples. In the case that r is greater than the number of initial abstractions of N we will replace N by the n.f. N' η-convertible to N and with r initial abstractions. This can be done since $N\ X_1 \dots X_r$ and $N'\ X_1 \dots X_r$ are β-convertible and, moreover, $\mathcal{S}[N,r] = \mathcal{S}[N',r] = \mathcal{S}[N']$ by definition 3.
Let x,y be the variables of an arbitrary d-couple and $(h, \mu ; k, \nu)$ be the structure of this d-couple. Moreover let a be a variable which don't occur free in N and which is free for x and y in N. Since $X_1,\dots,X_r \in \mathcal{C}[N,r]$, X_h or X_k must satisfy conditions 1 or 2 of definition 7 for $(h, \mu ; k, \nu)$. We split the proof according to these four possible cases:

i) $X_h \in \mathcal{N}_\omega$
ii) $X_k \in \mathcal{N}_\omega$
iii) X_h satisfies condition 1 and $\mu \neq \varepsilon$
iv) X_k satisfies condition 2 and $\nu \neq \varepsilon$.

case i).In this case x is replaced by X_h. Let R_1 be the n.f. obtained by replacing in N the occurrence of x which belongs to the considered d-couple by a. R_1 contains at most u d-couples. Therefore by inductive hypothesis $R_1\ X_1 \dots X_r \geq R_1'$ which is in n.f..
By construction $N\ X_1 \dots X_r$ is convertible to $\lambda a R_1'\ X_h$. Since by hypothesis $X_h \in \mathcal{N}_\omega$, Lemma 1 assures us that $\lambda a\ R_1'\ X_h$ possesses n.f..
case ii). This case may be proved simply by rephrasing the proof of case i with :

- x replaced by y
- X_h replaced by X_k.

case iii). Let R_2 be the n.f. obtained by replacing the occurrence of x which belongs to the considered d-couple by ax. R_2 contains at most u d-couples. Therefore by inductive hypothesis: $R_2X_1\ldots X_r \geq R_2'$ which is in n.f.. By construction N $X_1\ldots X_r$ is convertible to $\lambda aR_2'\mathbf{I}$ (7). We show that $R_2'[a/\mathbf{I}]$ has n.f. by performing the β-reductions always from the innermost ones. In R_2' the subterms whose head variable is a (let them be s) are obviously in n.f. In $R_2'[a/\mathbf{I}]$ the first components of these subterms are convertible to the λ-terms which replace respectively the s occurrences of x in some contracta of N $X_1\ldots X_r$. We recall that, since no structure of $\mathcal{S}[X_h,t_h]$ matches with μ, Lemma 2 assures us that any n.f. Y that replaces x in some contractum of $NX_1\ldots X_r$ is such that $Y\in\mathcal{N}_\omega$. First we reduce the subterms of $R_2'[a/\mathbf{I}]$ which coincide with the innermost occurrences of $\mathbf{I}$ applied to a given number of n.f.s. The first arguments of these occurrences of $\mathbf{I}$ are n.f.s which replace x in some contracta of $NX_1\ldots X_r$ and then they belong to $\mathcal{N}_\omega$.This assures us that the current subterms possess n.f. and therefore we obtain a λ-term in which the only redexes are (as before) occurrences of $\mathbf{I}$ applied to n.f.s. Again the first arguments of these subterms are n.f.s that replace x in some contracta of $NX_1\ldots X_r$ and then they belong to $\mathcal{N}_\omega$. We can now iterate the same argument as before, reducing at each step the subterms which coincide with the innermost occurrences of $\mathbf{I}$, until we have exhausted them. (This process will surely stops since there is only a finite number of occurrences of $\mathbf{I}$ and the reduction strategy implies that no occurrence of $\mathbf{I}$ can be generated).

case iv). This case may be proved simply by rephrasing the proof of case iii with:

- x replaced by y
- X_h replaced by X_k
- t_h replaced by t_k.

□

Example 7. We apply the n.f. N of example 1 to the n.f.s:

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(7) $\mathbf{I} \equiv \lambda xx$.

$$X_1 = \lambda z_1 (z_1 \lambda z_2 (z_2 z_2))$$

and
$$X_2 = \lambda z_1 (z_1 \lambda z_2 \lambda z_3 \lambda z_4 \ (a_3 z_2) \lambda z_5 (a_4 (z_5 z_5))).$$

Since $\mathcal{S}[X_1, 0] = \Phi$ and $\mathcal{S}[X_2, 1] = \{(1,2,1;\ 1,2,1)\}$ by definition 7 we may assure that $X_1, X_2 \in \mathcal{C}[N,2]$. In fact:

$$N\ X_1 X_2 \geq a_1\ \lambda x_1\ (a_3 a_2)\ \lambda x_2 (a_4 (x_2 x_2)).$$

We notice that by means of the theory developed in [3] and [6] this result could not be proved.

5. Conclusion

In this paper some termination properties of applications of λ-terms in n.f. have been presented. We point out that through the set of n.f.s it is also possible to represent the set of all λ-terms.
To see this we state here, only sketching the proof, some elementary facts true for λ-terms.

i) Any $T \in \Lambda$ is either a n.f. or it is convertible to a finite combination of n.f.s .

ii) Any finite combination M of n.f.s is convertible to the form $N X_1 \ldots X_r$ where $N, X_1, \ldots, X_r$ are n.f.s for some $r > 0$.

The proof of i follows from an inside-outside iterated application of a basic theorem in combinatory logic [4] which allows to replace $\lambda x(F[x]\, G[x])$ by $\mathbf{S}\, \lambda x F[x] \lambda x G[x]$(8). With such a procedure all unwanted abstractions of applications of λ-terms can be eliminated. The proof of ii succeds by locating first all the distinguished head subterms $X_1, \ldots, X_r$ of M, replacing their occurrences by different variables, say $x_1, \ldots, x_r$, creating $M[x_1, \ldots, x_r]$ and defining $N \equiv \lambda x_1 \ldots \lambda x_r\, M[x_1, \ldots, x_r]$ which becomes a n.f. . Bringing together i and ii one has:

$$(\forall\ T \in \Lambda)(\exists\, r, N, X_1, \ldots, X_r \in \mathcal{N})\ \ [T = N\ X_1 \ldots X_r].$$

so that no much generality is lost by restricting the study of termination properties to λ-terms with the shape $N\ X_1 \ldots X_r$.

The algorithm to test if $X_1, \ldots, X_r \in \mathcal{C}[N,r]$ has been implemented with run time $O(|N|^2 \log_2 |N| + \sum_1^r{}_i\, |X_i|^2 \log_2 |X_i|)$.
The description of this implementation has been leaved out here for sake of brevity. Since it has been shown [7] that even in a restricted

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(8) $\mathbf{S} = \lambda x \lambda y \lambda z\ (xz(yz))$.

number of cases the number of reductions required to reach the n.f. of a λ-term is bound by a non-elementary primitive recursive function (of some numbers depending on the structure of this λ-term), the ap - lication of the termination test requires a number of steps negligible with respect to the reduction algorithm.

Appendix

Here we will prove the following more general property, of which Lemma 2 is a particular case, and whose proof is no more expensive.

Property. Let $N \equiv \lambda x_1 \ldots \lambda x_n (\mathfrak{z} N_1 \ldots N_m), X_1, \ldots, X_h$ be h+1 arbitrary n.f.s and x a variable whose a.p. in N is h, μ with $\mu \neq \varepsilon$. Let k be the order of x_h in N and u a positive integer. If x in any contractum of $NX_1 \ldots X_h$ is replaced by a n.f. Y and $\mathfrak{s}[Y]$ contains a structure matching with u then $\mathfrak{s}[X_h, k]$ must contain a structure matching with μ, u.

Proof. Before giving a formal proof of this Property we introduce the following definition: in a n.f. $N \equiv \lambda x_1 \ldots \lambda x_n (\mathfrak{z} N_1 \ldots N_m)$ we will say that all subterms of $\mathfrak{z} N_1 \ldots N_m$ occur in the body of x_l $(1 \leq l \leq n)$. We prove the Property by induction on $|N|$.

First step. $|N| = 1$ implies $N \equiv \lambda x_1 \ldots \lambda x_n \mathfrak{z}$. Then all variables of N have a.p. such that $\mu = \varepsilon$. The Property is vacuously true.

Indcutive step. We assume that the Property is true for $|N| \leq s$ and we prove it for $|N| = s+1$. Let be $\mu = q, p, \rho$ (where ρ is possibly empty). The variable x must occur in a component of N, say N_i $(1 \leq i \leq m)$. If x in $\lambda x_1 \ldots \lambda x_h N_i$ has still a.p. h, then, since $|\lambda x_1 \ldots \lambda x_h N_i| \leq s$, by inductive hypothesis the Property is true for $\lambda x_1 \ldots \lambda x_h N_i, X_1, \ldots, X_h$. Let be $\mathfrak{z}' \equiv X_j$ if $\mathfrak{z} \equiv x_j$ with $1 \leq j \leq h$ and $\mathfrak{z}' \equiv \mathfrak{z}$ otherwise. If N_l' for $1 \leq l \leq m$ is any contractum of $\lambda x_1 \ldots \lambda x_h N_l \; X_1 \ldots X_h$, then clearly N' $\lambda x_{h+1} \ldots \lambda x_n (\mathfrak{z}' N_1' \ldots N_m')$ is a contractum of $N X_1 \ldots X_h$. Then the Property is true also for $N, X_1 \ldots, X_h$ since:

- x in N_i' and in N' is replaced by the same λ-term
- the order k' of x_h in N_i is less than or equal to k and therefore $\mathfrak{s}[X_h, k'] \subseteq \mathfrak{s}[X_h, k]$.

Otherwise, if x in N_i will have a.p. p, ρ , then $\mathfrak{z} \equiv x_h$, q=i, $\mu = i, p, \rho$ and $k \geq m$, i. e. $k \geq i$.

We will show that there is no n.f. X_h such that both the following conditions hold:

- no structure of $\mathfrak{I}[X_h,k]$ matches with μ,u
- the variable x is replaced in some contractum of $N X_1 \ldots X_h$ by a n.f. Y such that $\mathfrak{I}[Y]$ contains a structure matching with u.

Let be $X_h \equiv \lambda z_1 \ldots \lambda z_g (\tau U_1 \ldots U_f)$. If $g < i$ and τ is free, then x cannot be replaced against the hypothesis. If $g < i$ and τ is bound with a. p. j', then (since $k \geq i$) $\mathfrak{I}[X_h,k]$ contains the structure (j';i) matching with μ,u against the hypothesis.

Then we must consider the case $g \geq i$. Since x is replaced only if the p-th initial abstraction of N_i' is reduced, z_i must occur in $\tau U_1 \ldots U_f$ as head variable of a subterm which satisfies at least one of the following conditions:

a) it is, in its turn, a component of a subterm whose head variable is replaceable (that is it has defined a.p.);

b) it has at least p components.

In fact, if all subterms of $\tau U_1 \ldots U_f$ don't satisfy neither condition a nor b, the p-th initial abstraction of N_i' will never be reduced. If condition a is verified, then $\tau U_1 \ldots U_f$ has a subterm of the shape $\partial R_1 \ldots R_v$ where ∂ has defined a.p., say ν, and the head variable of R_v is z_i. But in this case $(\nu;i) \in \mathfrak{I}[X_h,k]$ and i matches with μ,u against the hypothesis.

If condition b is satisfied $\tau U_1 \ldots U_f$ has a subterm of the shape $z_i Z_1 \ldots \ldots Z_p$. We must distinguish two further cases:

b_1) the head variable of Z_p is replaceable

b_2) the head variable of Z_p is non-replaceable.

In case b_1 let φ be the a.p. of the head variable of Z_p. Then $(i;\varphi) \in \in \mathfrak{I}[X_h,k]$ and i matches with μ,u against the hypothesis.

In case b_2 let be $\rho = i',p',\rho'$. We rephrase the same argument as before observing that x can be replaced only if Z_p has at least i' initial abstractions, i.e. $Z_p \equiv \lambda y_1 \ldots \lambda y_{i'} \bar{Z}_p$ and there exists at least one subterm of $\bar{Z}_p$ which satisfies conditions a or b, where z_i and p have been replaced respectively by $y_{i'}$ and p'. Really we enter an iterative procedure which may stop only on cases a or b_1 or when the a.p. is exhausted. In the last case let T be the subterm of X_h that we must consider. T occurs in the body of some abstractions: the first g of them are $\lambda z_1 \ldots \lambda z_g$. Let the next replaceable variables be $z_{g+1}, \ldots, z_{g+w}$ $(w \geq 0)$. Then x in any contractum of $N X_1 \ldots X_h$ will be replaced by the λ-term:

$V = \lambda z_1 \ldots \lambda z_{g+w} T \; N_1[x_1/X_1, \ldots, x_h/X_h] \ldots N_i \; [x_1/X_1, \ldots, x_h/X_h] R_1 \ldots R_{g-i+w}$ where the indexed R denote the λ-terms which replace $z_{i+1}, \ldots, z_{g+w}$ in the current contractum of $N X_1 \ldots X_h$. In particular if z_{i+e} would not be replaced we assume $R_e \equiv z_{i+e} (1 \leq e \leq g-i+w)$.

Since the head variable of T is by hypothesis non-replaceable, then the head variable of V is free. If V reduces to a n.f. Y, it is sufficient to prove that $\mathfrak{s}[Y]$ don't contain any structure matching with u.

If $\mathfrak{s}[X_h, k]$ don't contain a structure matching with μ,u then:

i) $\mathfrak{s}[T]$ don't contain any structure matching with u. In fact if $\mathfrak{s}[T]$ would contain a structure $(\zeta;\gamma)$ matching with u, by definition this structure should became $(\mu,\zeta;\mu,\gamma)$ in $\mathfrak{s}[X_h, k]$, against the hypothesis.

ii) $\mathfrak{s}[\lambda z_1 \ldots \lambda z_{g+w} T]$ don't contain any structure matching with u+g+w=t. In fact if $\mathfrak{s}[\lambda z_1 \ldots \lambda z_{g+w} T]$ would contain a structure $(t,\zeta;\gamma)$ matching with t, this structure should became $(\mu,u,\zeta;\psi)$ (9) in $\mathfrak{s}[X_h, k]$ against the hypothesis. We consider the set $\mathfrak{V}$ of variables which are t-replaceable in $\lambda z_1 \ldots \lambda z_{g+w} T$ with t = u+g+w (that is, they can be replaced only if the u-th initial abstraction of T is reduced). By condition ii) if $y \in \mathfrak{V}$ then y occurs in $\lambda z_1 \ldots \lambda z_{g+w} T$ as head variable of subterms such that:

- they are, in their turn, components of subterms whose head variables are non-replaceable
- all their components have non-replaceable head variables.

The non-replaceable variables of $\lambda z_1 \ldots \lambda z_{g+w} T$ remain non-replaceable in Y. Therefore also in Y the subterms whose head variable is y satisfy the former conditions and so $\mathfrak{s}[Y]$ cannot contain any structure matching with u.

□

(9) The relation between ψ and γ depends on γ according to the definition of a.p..

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