TIME OPTIMAL CONTROL PROBLEM FOR DIFFERENTIAL INCLUSIONS

V.I.Blagodatskih

Mathematical Institute of USSR Academy of Sciences,
Moscow, USSR

I. Introduction

Let E^n be Euclidean space of state-vectors $x=(x_1,\dots,x_n)$ with the norm $\|x\|=\sqrt{\sum_{i=1}^n x_i^2}$ and $\Omega(E^n)$ be metric space of all nonempty compact subsets of E^n with Hausdorff metric

$$h(F,G)=\min\{d: F\subset S_d(G),\ G\subset S_d(F)\}$$

where $S_d(M)$ denotes a d - neighborhood of a set M in the space E^n.

Let us consider an object with behaviour described by the differential inclusion

$$\dot{x}\in F(t,x) \tag{I}$$

where $F: E^1\times E^n \to \Omega(E^n)$ is a given mapping. The absolutely continuous function $x(t)$ is the solution of the inclusion (I) on the interval $[t_0,t_1]$, if the condition $\dot{x}(t)\in F(t,x(t))$ is valid almost everywhere on this interval.

On the one hand the differential inclusion is the extension of ordinary differential equations

$$\dot{x}=f(t,x) \tag{2}$$

when function $f(t,x)$ is multivalued. On the other hand this extension is not formal, for many different problems may be transformed to differential inclusions and the development of differential inclusion permits us to solve these problems. For example, A.F.Filippov with the help of differential inclusions investigated [I] the solutions of differential equation (2) on the sets where function $f(t,x)$ had discontinuities. N.N.Krasovski used the differential inclusion [2] for constructing a strategy in differential games. Let us consider the connection of differential inclusions with some other problems.

Optimal control problem was considered first [3] by L.S.Pontryagin and others for systems described by the equation

$$\dot{x}=f(t,x,u),\quad u\in V. \tag{3}$$

This problem may be transformed to determination of optimal solution $x(t)$ of the differential inclusion

$$\dot{x}\in f(t,x,V)=\{f(t,x,u): u\in V\}.$$

Knowing the optimal solution $x(t)$ it is possible with help of Filippov's implicit functions lemma [4] to construct for system (3) a control $u(t)$ which produces this optimal solution. Note that control system (3) may be transformed to differential inclusion form even in the case when the set V depends on time and state, i.e. is of the form $V(t,x)$. On the other hand inclusion (I) may be considered as a control system with changing control domain

$$\dot{x}=v,\quad v\in F(t,x).$$

It will be noted that optimal control problems stimulated very much the development of the theory of differential inclusions.
The implicit differential equation

$$f(t, x, \dot{x}) = 0$$

may be transformed to a differential inclusion form, too,

$$\dot{x} \in F(t,x) = \{v : f(t,x,v) = 0\}.$$

On the other hand inclusion (I) may be considered as the implicit differential equation

$$\rho(\dot{x}, F(t,x)) = 0,$$

where $\rho(p, A)$ denotes the distance from a point p to a set A:

$$\rho(p, A) = \min_{a \in A} \| p - a \|.$$

A system of differential inequalities

$$f_i(t, x, \dot{x}) \leq 0, \quad i = 1, \dots, k,$$

may be transformed to the differential inclusion

$$\dot{x} \in F(t,x) = \{v : f_i(t,x,v) \leq 0, \ i = 1, \dots, k\}.$$

On the other hand inclusion (I) may be considered in the case of $F(t,x)$ convex as an infinite system of differential inequalities

$$(\dot{x}, \psi) \leq c(F(t,x), \psi), \quad \psi \in S_1(0),$$

where $c(F, \psi)$ is the support function of the set F:

$$c(F, \psi) = \max_{f \in F} (f, \psi).$$

2. Time optimal control problem

Let M_0, M_1 be nonempty closed subsets of E^n. The solution $x(t)$ given on the interval $[t_0, t_1]$ transfers M_0 to M_1 in time $t_1 - t_0$ if the conditions $x(t_0) \in M_0$, $x(t_1) \in M_1$ are satisfied. The time optimal control problem is to determine the solution of the inclusion (I) transferring the set M_0 to the set M_1 in a minimum time.

Maximum principle. Let the support function $c(F(t,x), \psi)$ of the inclusion (I) be continuously differentiable in x and the solution $x(t)$, $t_0 \leq t \leq t_1$, transfer the set M_0 to the set M_1. We shall say that the solution $x(t)$ satisfies the maximum principle on interval $[t_0, t_1]$ if there exists such nontrivial solution $\psi(t)$ of the adjoint system

$$\dot{\psi} = - \frac{\partial c(F(t, x(t)), \psi)}{\partial x} \tag{4}$$

that the following conditions are satisfied:

A) the maximum condition

$$(\dot{x}(t), \psi(t)) = c(F(t, x(t)), \psi(t))$$

is satisfied almost everywhere on the interval $[t_0, t_1]$;

B) the transversality condition on the set M_0: vector $\psi(t_0)$ is the support vector for the set M_0 at the point $x(t_0)$, that is

$$c(M_0, \psi(t_0)) = (x(t_0), \psi(t_0));$$

C) the transversality condition on the set M_1: vector $-\psi(t_1)$ is the support vector for the set M_1 at the point $x(t_1)$ that is

$$c(M_1, -\psi(t_1)) = (x(t_1), -\psi(t_1)).$$

3. Necessary conditions of optimality

Multivalued function $F: E^1 \times E^n \to \Omega(E^n)$ is called measurable if for any closed set $P \subset E^n$ the set $\{x : F(x) \cap P \neq \emptyset\}$ is Lebesgue measurable. The continuity and lipschitzability of multivalued function $F(x)$ is defined in the usual way. For example, the function $F(x)$ satisfies Lipschitz's condition with constant L if for any points $x, x' \in E^n$ the inequality

$$h(F(x), F(x')) \leqslant L \, \|x - x'\|$$

is valid. The number $|F| = h(\{0\}, F\}$ is called modulus of set F.

Theorem I. Let the multivalued function $F(t, x)$ of inclusion (I) be measurable in t and satisfy Lipschitz's condition in x with a summable constant $L(t)$ and $|F(t,x)| \leqslant g(t)$ where $g(t)$ is a summable function. Assume that the support function $c(F(t,x), \psi)$ is continuously differentiable in x, the sets M_0, M_1 are convex and solution $x(t)$, $t_0 \leqslant t \leqslant t_1$, transferring the set M_0 to the set M_1 is optimal. Then this solution satisfies maximum principle on the interval $[t_0, t_1]$. Moreover the condition

$$c(F(t_1, x(t_1)), \psi(t_1)) \geqslant 0$$

is valid.

The proof of the theorem I follows the plan suggested in the book [3] for systems of the type (3). The main difficulty is to define the variation of the solution. Here instead of the classical theorem on differentiability of solution with respect to initial condition (see, for example[5]) it's necessary to use theorem 2 stated below.

Let function $f: E^n \to E^1$ satisfy Lipshitz'a condition. The set of all partial limits of the gradient of this function at the point $x + h$ when $h \to 0$, that is

$$\partial f(x) = \overline{\lim_{h \to 0}} \; \nabla f(x+h)$$

is called the subdifferential of function $f(x)$ at the point x.

Theorem 2. Let $x(t)$, $t_0 \leqslant t \leqslant t_1$ be a solution of inclusion (I) with the initial condition x_0, $\psi(t)$ be a solution of the adjoint system (4) corresponding to $x(t)$ and $\delta x(t)$ be a solution of differential inclusion

$$\delta\dot{x} \in \partial_\psi \left[\frac{\partial c(F(t, x(t)), \psi(t))}{\partial x} \, \delta x \right]$$

with initial condition $\delta x(t_0) = h$, $\varepsilon > 0$. Then there exists such $y_\varepsilon(t)$ - solution of inclusion (I) with initial condition $y_\varepsilon(t_0) = x_0 + \varepsilon h$ defined on interval $[t_0, t_1]$ that the following condition is valid

$$y_\varepsilon(t) = x(t) + \varepsilon\,\delta x(t) + o(\varepsilon).$$

The idea of the proof is contained in paper [6].

Remark. The sets M_0, M_1 in theorem I may be nonconvex. It is sufficient that there exist the approximating cones to the sets M_0, M_1 at the points $x(t_0)$, $x(t_1)$, respectively. In this case the conditions B), C) in the maximum principle have to be replaced by conditions that the vectors $\psi(t_0), -\psi(t_1)$ are supports to respective approximating cones at the points $x(t_0), x(t_1)$.

4. Convexity of the set of solutions

Naturally the question arises: when is the maximum principle not only necessary but also sufficient condition of optimality? This seems to be very important to know when the set of solutions $\Sigma_{[t_0,t_1]}(P)$ of inclusion (I) with initial condition $x(t_0) \in P$ is convex in the space $C[t_0,t_1]$ of all continuous functions on the interval $[t_0, t_1]$. Denote by Z_τ the intersection of the set $\Sigma_{[t_0,t_1]}(P)$ by the plane $t=\tau$. The set $Z(\tau)$ is the set of all points at which it's possible to arrive along the solutions of the inclusion (I) from initial set P at a moment τ.

Multivalued function $F(t, x)$ is concave in x on the set M if the condition

$$\alpha F(t,x) + \beta F(t,x') \subset F(t, \alpha x + \beta x')$$

is valid for any points $x, x' \in M$ and for any numbers $\alpha, \beta \geq 0, \alpha+\beta=1$.

Theorem 3. Assume the initial set P be convex. Then the set of solutions $\Sigma_{[t_0,t_1]}(P)$ is convex if multivalued function $F(t,x)$ is concave in x on the sets $Z(\tau)$ for any $\tau \in [t_0, t_1]$.

The idea of the proof is contained in paper [7] .

5. Sufficient conditions of optimality

For the maximum principle to be also a sufficient condition of optimality of the solution $x(t)$ one needs to put two additional conditions on the solution $x(t)$.

We shall say that support function $c(F(t,x), \psi)$ is concave in x at the point x_0 in the direction ψ_0 if the condition

$$\left(\frac{\partial c(F(t,x_0),\psi_0)}{\partial x}, x - x_0\right) \geqslant c(F(t,x),\psi_0) - c(F(t,x_0),\psi_0)$$

is valid for any point $x \in E^n$.

Note that the concavity of a multivalued function $F(t,x)$ yields the concavity of the support function $c(F(t,x),\psi)$ at any point and in any direction.

Let $x(t)$, $t_0 \leqslant t \leqslant t_1$ be a solution of inclusion (I) and $\psi(t)$ be a respective solution of the adjoint system (4). We shall say that the solution $x(t)$ satisfies strong transversality condition on the set M_1 with the adjoint function $\psi(t)$ if the condition

$$c(M_1, -\psi(t)) < (x(t), -\psi(t))$$

is valid for any moment t: $t_0 \leqslant t < t_1$.

Theorem 4. Assume that M_0, M_1 are nonempty closed subsets of E^n, the solution $x(t)$ of inclusion (I) transfers the set M_0 to the set M_1 on the interval $[t_0, t_1]$ and satisfies the maximum principle on this interval and $\psi(t)$ is the respective solution of adjoint system (4). Assume that the support function $c(F(t,x),\psi)$ is concave in x at point $x(t)$ in direction $\psi(t)$ for any $t \in [t_0, t_1]$ and the solution $x(t)$ satisfies the strong transversality condition on the set M_1 with adjoint function $\psi(t)$. Then the solution is optimal.

The proof is contained in paper[8].

6. Uniqueness of optimal solution

In the case of continuous differentiability of the support function $c(F(t,x),\psi)$ in ψ the maximum condition A) and the adjoint system (4) may be written as the system of differential equations

$$\dot{x}(t) = \frac{\partial c(F(t,x(t)), \psi(t))}{\partial \psi}$$

$$\dot{\psi}(t) = -\frac{\partial c(F(t,x(t)), \psi(t))}{\partial x}.$$

Initial conditions for solution $(x(t), \psi(t))$ of this system may be determined from conditions B), C) of the maximum principle and from inclusions $x(t_0) \in M_0$, $x(t_1) \in M_1$. The question arises naturally in this case: when is for given initial conditions the unique solution $(x(t), \psi(t))$ determined from the maximum principle? The following theorem is an answer to this question.

Theorem 5. Assume that the support function $c(F(t,x),\psi)$ is measurable in t, continuously differentiable in (x,ψ) except for the points $\psi = 0$ and derivatives $c'_x(F(t,x),\psi)$, $c'_\psi(F(t,x),\psi)$ satisfy Lipschitz's condition in (x,ψ). Then for any given initial conditions for functions $x(t)$, $\psi(t)$ the optimal solution is unique.

7. Concluding remarks

Pontryagin's maximum principle was proved [3] for control system (3) for function $f(t,x,u)$ continuously differentiable in x. These systems may be transformed in differential inclusion form and support function in this case is

$$c(F(t,x),\psi) = \max_{u \in U} (f(t,x,u), \psi).$$

Theorem I is applied when this support function is continuously differentiable in x. It is not very difficult to show that there exists the function $f(t,x,u)$ continuously differentiable in x but for which the corresponding support function $c(F(t,x),\psi)$ does not have this property and vice versa. That is Pontryagin's maximum principle and theorem I, are intersected over some class of control systems of type (3). It is possible to formulate as a hypothesis the following theorem which includes the theorem I and Pontryagin's maximum principle.

Theorem 6. Suppose that the multivalued function $F(t,x)$ for inclusion (I) is measurable in t and satisfies Lipschitz's condition in x, sets M_0, M_1 are convex and solution $x(t)$, $t_0 \leq t \leq t_1$, is optimal. Then there exists such nontrivial absolutely continuous function $\psi(t)$ that the following conditions are valid.

A) $$\frac{d}{dt}\begin{pmatrix} x(t) \\ \psi(t) \end{pmatrix} \in \partial c_{(x,\psi)}\left(F(t, x(t)), \psi(t)\right)$$

almost everywhere on the interval $[t_0, t_1]$;

B) $$c(M_0, \psi(t_0)) = (x(t_0), \psi(t_0));$$

C) $$c(M_1, -\psi(t_1)) = (x(t_1), -\psi(t_1)).$$

References

I. A.F.Filippov, Differential equations with discontinuous right-hand side (Russian) , Matem. sbornik, 5I, N I, I960.

2. N.N.Krasovski, Game problems on encounter of motions. (Russian), Nauka, 1970.

3. L.S.Pontryagin et al., Mathematical theory of optimal processes (Russian) , Nauka, I96I.

4. M.Q.Jacobs, Remarks on some recent extensions of Filippov's implicit functions lemma, SIAM, Control, 5,N 4, I967.

5. L.S.Pontryagin, Ordinary differential equations (Russian) , Nauka, I970.

6. V.I.Blagodatskih, On differentiability of solutions with respect to initial conditions (Russian), Different. Uravn., 9, N I2, I973.

7. V.I.Blagodatskih, On convexity of domains of reachability (Russian), Different. Uravn., 8, N I2, I972.

8. V.I.Blagodatskih, Sufficient conditions of optimality for differential inclusions (Russian), Izv. AN SSSR, ser. Mathemat., N 3, I974.