Ω-OL SYSTEMS*

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Definition 1.1.1

An operator domain is a set Ω with a mapping $a : \Omega \to N$; the elements of Ω are called operators, and if $\omega \varepsilon \Omega$, then $a(\omega)$ is called the arity of ω. If $a(\omega) = n$, we say that ω is an n-ary operator. We write $\Omega(n) = \{\omega \varepsilon \Omega | a(\omega) = n\}$.

Definition 1.1.2

Let A be a set and Ω an operator domain, then an Ω-algebra structure on A is a family of mappings $\Omega(n) \to A^{A^n}$, $n \varepsilon N$. Thus with each $\omega \varepsilon \Omega(n)$ we associate an n-ary operation on A.

$$\omega : A^n \to A$$

Definition 1.1.3

The set A with an Ω-algebra structure on A is called an Ω-algebra and is denoted by (A,Ω) or A_Ω.
A is called the carrier of A_Ω.

Definition 1.1.4

For any Ω-algebra A_Ω and any $\omega \varepsilon \Omega(n)$, the application of ω to an n tupple $(a_1,a_2,\ldots,a_n)$ from A gives an element of A. We write this element in postfix Polish notation $a_1\ a_2\ \ldots\ a_n\omega$. If n=0, then this means that $\omega \varepsilon A$. These ω's are called constant operators.

*An extract from the author's thesis:
"ON THE ALGEBRAIC FOUNDATIONS OF DEVELOPMENTAL SYSTEMS".

In what follows, we assume a fixed Ω structure throughout, and will omit sometimes the subscript Ω.

Definition 1.1.5

Given Ω-algebras A_Ω and B_Ω, a mapping $f : A \to B$ and $\omega \varepsilon \Omega(n)$, we say f is compatible with ω, if for all $a_1, \ldots, a_n \varepsilon A$

$$f(a_1)f(a_2)\ldots f(a_n)\omega = f(a_1a_2\ldots a_n\omega).$$

If f is compatible with each $\omega \varepsilon \Omega$, then f is said to be a homomorphism from A to B.

Definition 1.1.6

Given any two Ω-algebras A_Ω and B_Ω, we say that B_Ω is a sub-algebra of A_Ω if $B \subset A$ i.e., if the carrier of B_Ω is a subset of the carrier of A_Ω.

Definition 1.1.7

Given a family $(A_{i\Omega})i\varepsilon I$ of Ω-algebras, $\prod_{i\varepsilon I} A_{i\Omega}$, the associated direct product is defined as follows.

Let P be the Cartesian product of the A_i's with projections $\Pi_i : P \to A_i$, then any element $p \varepsilon P$ is uniquely determined by its components $\Pi_i(p)$ and any choice of elements $(a_i \varepsilon A_i)_{i\varepsilon I}$ defines uniquely an element $p \varepsilon P$ by $\Pi_i(p) = a_i$ for all $i\varepsilon I$. Consequently if $p_1, p_2, \ldots, p_n \varepsilon P$ and $\omega\varepsilon\Omega(n)$, we can define $p_1p_2\ldots p_n\omega$ by the formuli

$$\Pi_i(p_1p_2\ldots p_n\omega) = \Pi_i(p_1)\Pi_i(p_2)\ldots\Pi_i(p_n)\omega \text{ for all } i \in I$$

This procedure defines an Ω structure on P simply by doing all operations componentwise. Consequently we have

Theorem 1.1.7

Direct product of Ω-algebras is an Ω-algebra. We note that the A_i's need not be distinct and that the projections are homomorphisms.

Definition 1.1.8

Let Ω be an operator domain and let $\overline{\Omega}$ be a shadow alphabet representing the operations of Ω, let X be an auxiliary alphabet disjoint from $\overline{\Omega}$. X is called the alphabet of free variables. We define the language of the Ω-words, $W_\Omega(X)$, as the following subset of $(X \cup \overline{\Omega})^*$

1. If $\omega \varepsilon \Omega(0)$, then $\overline{\omega} \varepsilon W_\Omega(X)$.

2. If $x \varepsilon X$, then $x \varepsilon W_\Omega(X)$.

3. If $a_1, a_2, \ldots, a_n \varepsilon W_\Omega(X)$ and $\omega \varepsilon \Omega(n)$, then $a_1a_2\ldots a_n\overline{\omega} \varepsilon W_\Omega(X)$.

4. Nothing else belongs to $W_\Omega(X)$ unless its being so follows from a finite number of applications of the set of rules 1,2 and 3.

Definition 1.1.9

Let X and $\bar{\Omega}$ be given as in Definition 1.1.8, then we define an Ω-algebra $(\Omega,(X \cup \bar{\Omega})^*)$ as follows
for any $a_1,a_2,\ldots,a_n \varepsilon (X \cup \bar{\Omega})^*$ and any $\omega \varepsilon \Omega(n)$
$\omega(a_1,a_2,\ldots,a_n) = a_1a_2\ldots a_n\bar{\omega}$.

Theorem 1.1.10

The Ω-words $W_\Omega(X)$ for a given set X, form an Ω-algebra the subalgebra of $(\Omega,(X \cup \bar{\Omega})^*)$ generated by X.

Proof:

By Definition 1.1.8, the language of Ω-words is closed under the Ω operations and is generated by X.
Consistent with this theorem, we have the following.

Definition 1.1.11

The Ω-algebra of the set of Ω-words $W_\Omega(X)$ is called an Ω-word algebra.

Definition 1.1.12

Given the set of Ω-words $W_\Omega(X)$, we define the following relation $<\Omega<$ over $W_\Omega(X)$, called the Ω-subword relation,

1) For all $n \varepsilon N$ and all $a_i \varepsilon W\ (X)$, $1 \leq i \leq n$, if $\omega \varepsilon \Omega(n)$, then $a_i <\Omega< b$ iff $a_1a_2\ldots a_n\omega = b$.

Furthermore, 2) for all $a,b,c \in W_\Omega(X)$

$a <\Omega< a$ and

3) if $a <\Omega< b$ and $b <\Omega< c$, then $a <\Omega< c$.

i.e., $<\Omega<$ is the reflexive transitive closure of the relation defined by 1).

If $a <\Omega< b$, then we say that a is an Ω-subword of b.

Definition 1.1.12.1

We define a mapping $h : W_\Omega(X) \to N$ as follows

$h(\omega) = 0$ for each $\omega \in \Omega(0)$

$h(x) = 0$ for each $x \in X$

$h(a_1a_2...a_n\omega) = 1+\max \{h(a_i) \mid 1 \leq i \leq n\}$

where $\omega \in \Omega(n)$ and $a \in W_\Omega(X)$.

We call h(a) the <u>height</u> of the Ω-word a.

These definitions allow us to represent Ω-words as ordered, labelled rooted trees, the free variables and nullary operators are the leaves of the tree, the other Ω-operators are the labels of the internal nodes. If $\omega \in \Omega(n)$, then n branches are eminating from the node labelled by ω. The right-most operator labels the root of the tree, and the distinction between free variables and nullary operators is only nominal. The reader is encouraged to draw Ω-words as trees in the examples.

Definition 1.1.12.2

Let v be an Ω-word of $W_\Omega(X)$, then we define a correspondence ℓ_v from the Ω-subwords of v into the non-negative integers as follows
$\ell_v(v) = 0$, and for any Ω-subwords b and a_i; $b, a_i < \Omega < v$,
$\ell_v(a_i) = 1+\ell_v(b)$ if $a_1 a_2 .. a_n \omega = b$, for all i, $1 \leq i \leq n$, and all $\omega \varepsilon \Omega(n)$.

We note that ℓ_v is not necessarily functional, since different occurrences of the same Ω-subword may have different values. We say that a certain occurrence of an Ω-subword u of v is on level n if $\ell_v(u) = n$.

The following standard theorems are stated without proof.

The reader is encouraged to consult any of the following excellent introductory reference texts:

A.G. KUROSH, Lectures on General Algebra, Chelsea (1963, 1965).

P.M. COHN, Universal Algebra, Harper-Row (1965).

Our notational conventions are chosen in attempt to parallel Cohn's symbolism, whose work is remarkably impressive for clarity and consistency.

Theorem 1.1.13

For any X and Y the corresponding Ω-algebras are isomorphic, $W_\Omega(X) \approx W_\Omega(Y)$, iff the cardinalities of X and Y are the same.

Theorem 1.1.14

The composition of homomorphisms Ω-algebras $f : A_\Omega \to B_\Omega$ and $g : B_\Omega \to C_\Omega$ is a homomorphism $f \circ g : A_\Omega \to C_\Omega$.

Theory 1.1.15

The image of a homomorphism $f : A_\Omega \to B_\Omega$ is a Ω-subalgebra of B_Ω.

Definition 1.1.16

An equivalence relation on an Ω-algebra A_Ω which is also a subalgebra of $A_\Omega \times A_\Omega$ is called a congruence relation on A_Ω.

Theorem 1.1.17

Let A be an Ω-algebra and ρ a congruence relation on A_Ω, then there is a homomorphism $\text{nat}\rho : A_\Omega \to (A/\rho, \Omega)$.

Definition 1.1.18

The Ω-algebra $(A/\rho, \Omega)$ is called the quotient algebra of A_Ω by ρ.

Theorem 1.1.19

For any Ω-algebras A,B and any generating set X of A, a homomorphism h of A into B is uniquely determined by its restriction $h|X$.

Theorem 1.1.20

Let A_Ω be any Ω-algebra and X any set, then any mapping $\theta : X \to A$ may be uniquely extended to a homomorphism $\overline{\theta} : W_\Omega(X) \to A_\Omega$.

Theorem 1.1.21

Any Ω-algebra A is a homomorphic image of an Ω-word algebra $W_\Omega(X)$ for some set X.

Definition 1.1.22

Γ, a symmetric set of designated pairs of Ω-words in the Ω-word algebra $W_\Omega(Y)$, is called a set of identical relations that hold in $W_\Omega(X)$.

$\Gamma \subset W_\Omega(Y) \times W_\Omega(Y)$, $(w_1,w_2) \in \Gamma$ is written as $w_1=w_2$.

Two Ω-words v and u in $W_\Omega(X)$ are called equivalent with respect to Γ, $v =\Gamma= u$, iff there is a finite sequence of transformations $v_i \to v_{i+1}$ $i=1,\ldots,n-1$: $v=v_1$ and $u=v$ such that there are Ω-words p_i and q_i, Ω-subwords of v_i and v_{i+1} respectively, $p_i <\Omega< v_i$ and $q_i <\Omega< v_{i+1}$, and v_{i+1} is obtained from v_i by replacing p_i by q_i, furthermore, p_i and q_i are the homomorphic images of w_1 and w_2 respectively under an arbitrary map $h_i : Y \to W_\Omega(X)$ extended to a homorphism, where $(w_1,w_2) \in \Gamma$.

Theorem 1.1.23

For any Ω-word algebra $W_\Omega(X)$ and any set of identical relations $\Gamma \subset W_\Omega(X) \times W_\Omega(X)$, the relation $=\Gamma=$ is a congruence relation on $W_\Omega(X)$.

Proof:

Obviously $=\Gamma=$ is an equivalence relation, but since the relation $<\Omega<$ is transitive and since subwords may be replaced by equivalent subwords, it is also a congruence relation. In fact, let $a_i =\Gamma= b_i$ for all i, $1\leq i\leq n$, then to show that for any $\omega\in\Omega(n)$, $a_1\ldots a_n\omega =\Gamma= b_1\ldots b_n\omega$, we are required to show that there is a finite sequence of transformations from $a_1\ldots a_n\omega$ to $b_1\ldots b_n\omega$. Suppose the length of the sequence of transformation from a_i to b_i is m_i, then the length of the required sequence of transformations is at most $\sum_{i=1}^{n} m_i$.

Definition 1.1.24

$W(\Omega,X,\Gamma)$ which denotes the factor algebra $(W_\Omega(X)/{=\Gamma=},\Omega)$ (or any algebra isomorphic to it), is called the free Ω-algebra of the variety Γ generated by X.

Note the generators of this algebra are the sets of $=\Gamma=$ equivalent words, equivalent to the elements of X.

In practice, when Γ is understood, $=\Gamma=$ is written simply as $=$.

Example 1.1.25

Consider the following free Ω-algebra Gd_1, $Gd_1=W(\Omega,X,\Gamma)$, where

$$\Omega(i) = \phi \quad \text{for all } i\neq 2$$
$$\Omega(2) = \{o\}$$
$$X = \{a\} \quad \text{and } \Gamma=\phi$$

Gd_1 is called the free groupoid generated by a singleton.

The following are the first few elements of Gd_1 ordered lexicographically, within length and height.

a

aao

aaoao

aaaoo

aaoaaoo

aaoaoao

aaaoaoo

aaaaooo

Example 1.1.26

Consider the following free Ω-algebra Sg_1.

$Sg_1 = W(\Omega, X, \Gamma)$,

where

$$\Omega(i) = \phi \quad \text{for all } i \neq 2$$

$$\Omega(2) = \{o\}$$

$$X = \{a\} \text{ and } \Gamma = \{xyozo = xyzoo\} \quad .$$

Sg_1 is called the free semigroup generated by a singleton. The following are the first few elements of Sg_1 in normal form ordered by length. By normal form we mean here the highest form in the lexicographical order among equivalent Ω-words representing an element of Sg_1.

a

aao

aaaoo

aaaaooo

We may note that aaaoo = aaoao are equivalent forms and so are

aaaaooo = aaaoaoo = aaoaoao = aaoaaoo.

To show the equivalence of the first and last elements in the chain, we use the map h, $h(x) = a, h(y) = a, h(z) = aao$.

Example 1.1.27

Consider the following free Ω-algebra G_{p_2} .

$G_{p_2} = W(\Omega, X, \Gamma)$, where

$\Omega(0) = \{1\}$

$\Omega(1) = \{^{-1}\}$

$\Omega(2) = \{o\}$

$\Omega(i) = \phi \qquad i>2$

$X = \{a,b\}$

$\Gamma = \{xyozo = xyzoo,\ xx^{-1}o = 1,$
$x^{-1}xo = 1,\ x1o = x,\ 1xo = x,$
$x^{-1}y^{-1}o = yxo^{-1},\ x^{-1-1} = x\}$

G_{p_2} is the free group generated by two elements. Γ is not minimal, but a convenient set of identical relations for a free group.

The following are the first few elements of G_{p_2} in normal form ordered lexicographically within length.

$1,a,b,a^{-1},b^{-1},$

$aao,abo,bao,bbo,$

$ab^{-1}o,a^{-1}bo,ba^{-1}o,b^{-1}ao,$

$a^{-1}a^{-1}o,a^{-1}b^{-1}o,b^{-1}a^{-1}o,b^{-1}b^{-1}o,$

$aaaoo,\ aaboo,\ldots.$

By normal form we mean here the highest form in the lexicographical order among equivalent Ω-words of minimal length.

Definition 1.2.1

Given a set A, we consider the set of all finitary operations on A, denoted by $\alpha(A)$,

$$\alpha(A) = \bigcup_{n\varepsilon N} A^{A^n} = \bigcup_{n\varepsilon N} \alpha_n(A)$$

where the set of n-ary operations on A is denoted by

$$\alpha_n(A) \quad , \quad \alpha_n(A) = A^{A^n}.$$

We define the composition of $\omega_1,\omega_2,\ldots,\omega_m$ with ω as an n-ary operation, δ, for each $\omega_1,\omega_2,\ldots,\omega_m \ \varepsilon\ \alpha_n(A)$ and $\omega\ \varepsilon\ \alpha_m(A)$ as follows

$$\delta : A^n \to A$$

$$\delta(x) = (x\omega_1)(x\omega_2)\ldots(x\omega_m)\omega \quad \text{for each } x\ \varepsilon\ A^n.$$

Any ordered set of operations, whose arities obey the requirement for composition is said to be conforming. We define for all $n > 0$, n n-ary operations in $\alpha_n(A)$, namely

$\Pi_{n1},\Pi_{n2},\ldots,\Pi_{nn}$, where for any given i, $1 \leq i \leq n$

$$\Pi_{ni} : A^n \to A$$

$$\Pi_{ni}\ (a_1,a_2,\ldots,a_i,\ldots a_n) = a_i \qquad \text{for all } a_j\ \varepsilon\ A, \quad 1 \leq j \leq n.$$

Π_{ni} selects the i-th element in an n-tuple of elements of A.

Π_{ni} is called the <u>i-th component projection of an n-tuple</u>.

$\{\Pi_{ni} \mid 1 \leq i \leq n\}$ is called the set of n-ary projection operators.

A set β of operations on A is called a <u>closed set of operations on A</u>, or a <u>clone on A</u> in short, if and only if β contains the projection operators and the conforming ordered subsets of β are closed under composition.

<u>Example 1.2.2</u>

$\alpha(A)$ is a clone, $\alpha_n(A)$ is a clone.

<u>Definition 1.2.3</u>

Let A be an Ω-algebra, then the clone generated by the Ω-operators on A is called the <u>clone of action</u> of Ω on A.

<u>Theorem 1.2.4</u>

Let A be an Ω-algebra and β the clone of action of Ω on A, let $x \in A^n$ and β_n the set of n-ary operators in β, then $(\{x\omega \mid \omega \in \beta_n\}, \beta_n)$ is the Ω-subalgebra generated by the entries of x.

<u>Proof</u>:

The n-ary projection operators will provide the generators and the operations in β will provide the Ω-algebra structure, the composition of the n-ary projection operators with elements of β are the set of all n-ary operators in β.

We may see this in a different form in the following.

Theorem 1.2.5

Let Ω be an operator domain, let $n \in N^+$, let $W_\Omega(X)$ be an Ω-word algebra generated by $X = \{x_1, x_2, \ldots, x_n\}$, let β be the clone of action of Ω on X, let Π_n be the vector of n-ary projection operators on X, namely

$\Pi_n = (\Pi_{n1}, \Pi_{n2}, \ldots, \Pi_{nn})$; $\beta(\Pi_n)$ then

denotes the set of n-ary operations that can be obtained from Π_n by repeated compositions of the projections and the operations in Ω, furthermore, there is an isomorphism between the elements of $W_\Omega(X)$ and $\beta(\Pi_n)$.

Proof:

Let us define a mapping ϕ from the elements of $W_\Omega(X)$ to the operations in $\beta(\Pi_n)$, and show that this mapping is an isomorphism.

1) $\phi(x_i) = \Pi_{ni} = \Pi_{ni}(\Pi_n)$.

Since the arity requirement for composition of operators in Ω are the same as the arity requirement for forming Ω-words, we may proceed inductively on the height of Ω-words as follows:

Suppose ϕ is defined for Y_k, the Ω-words of height less than k, so that $\phi(y) \varepsilon \beta(\Pi_n)$ for each $y \varepsilon Y_k$. Consider an Ω-word x of height equal to k such as $x = y_1 y_2 \ldots y_m \omega$, where $\omega \varepsilon \Omega(m)$. The y_i's are of height less than k, therefore, by hypothesis, each $\phi(y_i) \varepsilon \beta(\Pi_n)$ and $\phi(y_1)$, $\phi(y_2), \ldots, \phi(y_m)$ with ω form a conforming ordered subset of operations, therefore we may define in Π_n

2) $\quad \phi(x) = \omega(\phi(y_1), \phi(y_2), \ldots, \phi(y_m))$.

Since 1) defines ϕ for Ω-words of height 0, the induction is complete.

From 1) and 2) it is clear that ϕ is a homomorphism. Clearly every Ω operation can uniquely be so simulated and the composition of functions with a projection operator will not produce an additional function therefore ϕ is an isomorphism.

The following example is included simply to show an instance of the isomorphism and to make the notation more familiar.

Example 1.2.5

Let an Ω-word algebra $W_\Omega(X)$ be given as follows:

$X \quad = \{a,b\}$

$\Omega(0) = \phi$

$\Omega(1) = \{*, ^{-1}\}$

$\Omega(2) = \{o\}$

$\Omega(i) = \phi \qquad$ for $i>2$

Let $\Pi_{21}(a,b) = a$ and $\Pi_{22}(a,b) = b$.

Consider the Ω-word

$x = abo^{-1}a{*}o.$

$$\phi(x) = \phi(abo^{-1}a{*}o) = o(\phi(abo^{-1}), \ \phi(a*)) =$$

$$= o(^{-1}(\phi(abo)), \ *(\phi(a))) =$$

$$= o(^{-1}(o(\phi(a), \ \phi(b)), \ *(\Pi_{21}(a,b))) =$$

$$= o(^{-1}(o(\Pi_{21}(a,b), \ \Pi_{22}(a,b)), *(\Pi_{21}(a,b)))$$

A few remarks at this point are in order. First of all, in Theorem 1.2.5 the concept of an isomorphism is used in the relaxed sense that the parsing tree of the Ω-word x is graph theoretically isomorphic to the composition tree of the operators in $\phi(x)$, as labeled ordered rooted trees.

Secondly, the operation $\phi(x)$ is an n-ary operation from $(X \cup \Omega)^{*n} \to (X \cup \Omega)^*$, where $X = \{x_1, x_2, \ldots, x_n\}$.

Thirdly, Theorem 1.2.5 is simply the formal notion of the operation of substitution for the free variables in a given Ω-word.

On the basis of these remarks, we adopt the following:

<u>Definition 1.2.6</u>

Let x be an Ω-word in the Ω-word algebra $W_\Omega(X)$, where $X = \{x_1, x_2, \ldots, x_n\}$. We denote the value of the n-ary operation $\phi(x)$, $\phi(x) : (X \cup \Omega)^{*n} \to (X \cup \Omega)^*$, given in the proof of Theorem 1.2.5 as

$x(y_1, y_2, \ldots, y_n)$ for any n-tuple $(y_1, \ldots, y_n)$ of strings in $(X \cup \Omega)^*$ and any $x \in W_\Omega(X)$. This convention identifies the word x with the operation it represents as a substitution formula.

Example 1.2.6

Following Example 1.2.5, we obtain

$$x = abo^{-1}a*o$$

$x(a,b) \quad = abo^{-1}a*o$

$x(a,a^{-1}) \quad = aa^{-1}o^{-1}a*o$

$x(a^{-1},abo) = a^{-1}aboo^{-1}a^{-1}*o$

$x(b,oa) \quad = boao^{-1}b*o$

$abo^{-1}a*o(b,a) = bao^{-1}b*o$

We now introduce operations that are homomorphic images of compositions of operations in an Ω-word algebra.

To define operations of arbitrary arity, we will need a potentially infinite alphabet for the free substitutional variables. To specialize an operation, we use the scheme of explicit transformations, i.e., we permute or identify variables, put constants for variables and add new variables, all this may be accomplished by composition with the projection operators and substitution. Formally the development is given as follows:

Definition 1.2.7

Given an infinite set $X = \{x_1, x_2, x_3, \ldots\}$, consider the sequence of ordered subsets of S.

$X_1 = (x_1)$, $X_2 = (x_1, x_2), \ldots, X_n = (x_1, x_2, \ldots, x_n), \ldots$

let Ω be a operator domain, let the corresponding Ω-word algebras be the sequence

$W_\Omega(X_1)$, $W_\Omega(X_2), \ldots, W_\Omega(X_n), \ldots$

let x be a given Ω-word in $W_\Omega(X_n)$ with the correspondingly denoted operation $x(x_1, x_2, \ldots, x_n) = x$, which is a homomorphism

$$x : W_\Omega(X_n)^n \to W_\Omega(X)$$

let Ω' be an operator domain and let d be a given arity preserving mapping, i.e.

$$d : \Omega \to \Omega'$$

such that

$$d[\omega] = \omega' \quad \text{implies} \quad a(\omega) = a(\omega') ,$$

let $A_{\Omega'}$ be an Ω'-algebra, and for any

$$a_1, a_2, \ldots, a_n \in A_{\Omega'} \quad ,$$

let h be the homomorphic extension of the map $h(x_i) = a_i$,

then we define the operation $\tilde{x}$ as follows

$$\tilde{x} : A^n \to A$$

$$\tilde{x}(a_1,a_2,\ldots,a_n) = d[x](h(x_1),h(x_2),\ldots,h(x_n))$$

(In practice d is bijective and Ω and Ω' may be identified). The definition of $\tilde{x}$ may be exhibited by the following commuting diagram of homomorphisms.

$$\begin{array}{ccc} W_\Omega(X_n)^n & \xrightarrow{h} & A^n \\ x \downarrow & & \downarrow \tilde{x} \\ W_\Omega(X_n) & \xrightarrow[h]{} & A \end{array}$$

$\tilde{x}$ is called a <u>principal derived n-ary operation</u> on $A_{\Omega'}$, and is denoted by the expression $\tilde{x}(x_1,x_2,\ldots,x_n)$.

For any integer $k \leq n$ and any set of elements $b_1,b_2,\ldots,b_k \in A$, the operation $\tilde{x}(x_1,x_2,\ldots,x_m,b_1,b_2,\ldots,b_k)$, where $m=n-k$, is called a derived n-ary operation on $A_{\Omega'}$. In this definition we chose to specialize the last k arguments for notational convenience only.

<u>Example 1.2.7</u>

Consider the Ω_1-algebra given by

$A_{\Omega_1} = (\{1,a,b\} \;, \; \{o,1,^{-1}\})$, where $a(o) = 2$, $a(1) = 0$, $a(^{-1}) = 1$

o	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

	1	a	b
$^{-1}$	1	b	a

It is easy to see that A_{Ω_1} is the cyclic group on $\{1,a,b\}$. Consider the subgroup closure operator

$$\kappa(x,y) = x^{-1}yo(x,y) = o(^{-1}(\Pi_x(x,y)),\Pi_x(x,y)),$$

where

$$\Pi_x(x,y) = x \quad \text{and} \quad \Pi_y(x,y) = y.$$

The operator κ is a principal derived binary operator on A_{Ω_1}.

<u>Example 1.2.8</u>

On the other hand, consider the Ω_2-algebra, given by $A_{\Omega_2} = (\{1,a,b\}, \{\kappa\})$, where $a(\kappa) = 2$ and the multiplication table for κ is given as derived in the previous example, then consider the operations $\underline{o}$, $\underline{1}$, and $^{-1}$, where

$$\underline{1}(x,y) = \kappa(\Pi_x(x,y) , \Pi_x(x,y))$$

$$\underline{^{-1}}(x,y) = \kappa(\Pi_x(x,y) , \underline{1}(x,y)) \qquad \text{and}$$

$$\underline{o}(x,y) = \kappa(^{-1}(x,y) , \Pi_y(x,y)) .$$

The operations $\underline{1}(x,y)$, $\underline{}^{-1}(x,y)$ and $o(x,y)$ are principal derived binary operations on A_{Ω_2}.

$\underline{1}$	1	a	b
1	1	1	1
a	1	1	1
b	1	1	1

$\underline{}^{-1}$	1	a	b
1	1	1	1
a	b	b	b
b	a	a	a

$\underline{o}$	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

and clearly they are respectively equivalent to the operations 1, $^{-1}$ and o in A_Ω in the previous example, in fact, they are given as the following derived operations:

$1 = \kappa(\underline{1},1)$, $^{-1}(x) = \kappa(x,1)$ and $o(x,y) = \underline{o}(x,y)$.

Definition 1.2.9

For an Ω-algebra A_Ω, the derived unary operations are called <u>derived translations on A_Ω</u>. If a derived unary operation $\underline{f}$ is principal, then $\underline{f}$ is called a <u>principal derived translation on A_Ω</u>. A subset S of A such that $S = \{a, f(a) , f(f(a)) , f(f(f(a))),\ldots\}$ is called a <u>splinter in A_Ω</u> or an <u>Ω-algebraic splinter in A</u> for any derived translation $\underline{f}$ and any element $\underline{a}$ in A. Splinters play an important role in the fundations of mathematics, and were studied by Ullian, Myhill, Young and Rogers; they consider $\underline{f}$ to be any recursive function. Splinters also turn up in disguise in the study of autonomous automata, in the study of cyclic submonoids of the monoid of endomorphisms of

Ω-algebras and in the study of fixed point theorems in general.

Definition 1.2.10

Let A_Ω be an Ω-algebra, we define the Ω'-algebra $\mathcal{B}(A)_{\Omega'}$ as follows. For any $\omega \in \Omega(n)$, we define ω' as follows

$$\omega' : \mathcal{B}(A)^n \to \mathcal{B}(A)$$

$$\omega'(X_1,X_2,\ldots,X_n) = \{\omega(x_1,x_2,\ldots,x_n) \mid x_i \varepsilon X_i \quad \text{for all } i,\ 1\underline{<}i\underline{<}n\} .$$

We identify Ω and Ω'.

$\mathcal{B}(A)_\Omega$ is called the Ω-set algebra on A. In $\mathcal{B}(A)_\Omega$ the presence of the binary operator $\cup$ is tacitly assumed. As an immediate consequence of the definition, we have the following:

Theorem 1.2.11

For any Ω-set algebra $\mathcal{B}(A)_\Omega$ and any $\omega \in \Omega(n)$

$$\omega(X_1,X_2,\ldots,X_i,\ldots,X_n) \cup \omega(X_1,X_2,\ldots,X_i',\ldots,X_n) =$$

$\omega(X_1,X_2,\ldots,X_i \cup X_i',\ldots,X_n)$ where the X_i and X_i' are subsets of A.

Proof:

The left-hand side and the right-hand side are same as sets.

Example 1.2.11

Consider the free monoid generated by a doubleton $\mathfrak{W}_2 = W(\Omega, Y, \Gamma)$, where $\Omega = \{\lambda, o\}$, $Y = \{a,b\}$

$a(\lambda) = 0$ and $a(o) = 2$, $\Gamma = \{xyozo = xyzoo,\ \lambda xo = x\lambda o\}$.

The first few normalized elements of $\mathfrak{W}_2$ in lexicographical order within length are

λ,a,b,aao,abo,bao,bbo,aaaoo,aaboo,abaoo,abboo,...

Here the binary operation $\underline{o}$ is called the string catenation and the following is an instance of that

$$o(aao,b) = aaobo = aaboo.$$

The operation symbol may be omitted in the forms. Consider the Ω-set algebra $\mathfrak{P}(\mathfrak{W}_2)_\Omega$ corresponding to $\mathfrak{W}_2$, namely the algebra of languages over the binary alphabet with operation set-catenation or set-product. The following is an instance of this operation:

$$o(\{a,aa\}, \{b,ab,aab\}) = \{ab,aab,aaab,aaaab\}.$$

Because of the previous theorem, we have the following identity for any set A,B,C,D in $\mathfrak{P}(\mathfrak{W}_2)_\Omega$

$$o(A \cup B,\ C \cup D) = o(A,C) \cup o(B,C) \cup o(A,D) \cup o(B,D).$$

A set valued set mapping f is called monotonic if it preserves the inclusion relation, i.e. $X \subset Y$ implies $f(X) \subset f(Y)$. Using this terminology, we can state Theorem 1.2.11 equivalently as follows:

Corollary 1.2.11

For any Ω-set algebra $\mathcal{B}(A)_\Omega$ any derived translation is monotonic.

Proof: Let $X \subset Y \subset A$ and without loss of generality assume that the derived translation f is given as follows

$f(V) = \omega(V, A_1, A_2, \ldots, A_{n-1})$ for all $V \subset A$, where $A_i \subseteq A$, $o<i<n$, and ω is in the clone of action of Ω such that $a(\omega) = n > o$, then $X \subset Y$ implies that $Y = X \cup Z$, where $Z \subset A$, furthermore by Theorem 1.2.11

$$f(Y) = f(X \cup Z) = \omega(X \cup Z, A_1, A_2, \ldots, A_{n-1}) =$$

$$\omega(X, A_1, A_2, \ldots, A_{n-1}) \cup \omega(Z, A_1, A_2, \ldots, A_{n-1}) = f(X) \cup f(Z),$$

therefore

$$f(X) \cup f(Y).$$

Corollary 1.2.12

For any Ω-set algebra $\mathcal{B}(A)_\Omega$ and any $\omega \in \Omega(n)$, the following inclusion holds:

$$\omega(X_1,X_2,\ldots,X_i,\ldots,X_n) \cap \omega(X_1,X_2,\ldots,X_i',\ldots,X_n) \supset$$

$$\omega(X_1,X_2,\ldots,X_i \cap X_i',\ldots,X_n) ,$$

where the X_i's and X_i' are subsets of A.

Proof:

The right-hand side is included in each term of the intersection on the left, by the previous theorem, consequently it is included in their intersection. We note that the inclusion is proper in general and that $\omega(X_1,\ldots,\phi,\ldots,X_n) = \phi$.

Furthermore, if $X \subset Y$, then the splinter $(Y,f(Y),f(f(Y)),\ldots)$ dominates the splinter $(X,f(X),f(f(X)),\ldots)$ as a set sequence.

Definition 1.2.13

Let f be a derived translation on A_Ω, let a be an element of A, then the mapping $S[f,a,A_\Omega]$ is called an f-splinter sequence on A_Ω generated by a.

$S[f,a,A_\Omega] : N \to A_\Omega$

$S[f,a,A_\Omega](o) = a$

$S[f,a,A_\Omega](n) = f(S[f,a,A_\Omega](n-1))$

In what follows, we generalize the concept of the splinter to direct powers of Ω-algebras. By direct power we mean a direct product in which all factors are identical. Thus

$$A^n_\Omega = \overset{1}{\tilde{A}}_\Omega \times \overset{2}{\tilde{A}}_\Omega \times \dots \times \overset{n}{\tilde{A}}_\Omega$$

Definition 1.2.14

τ, an n-tuple of derived n-ary operations on an Ω-algebra A_Ω is called a derived transformation on A^n_Ω. If all the derived n-ary operations are principal, then it is called a principal derived transformation on A^n_Ω.

Let $\tau = (t_1, t_2, \dots, t_n)$ and $t_i : A^n \to A$, $1 \le i \le n$, then $\tau : A^n \to A^n$.

Definition 1.2.15

Let τ be a derived transformation on A^m_Ω and $a \in A^m_\Omega$, then we define an m-ary splinter as the following set of elements of A^m

$\{a, \tau(a), \tau(\tau(a)), \tau(\tau(\tau(a))), \dots\}$

$S[A^m_\Omega, \tau, a]$ will denote the following mapping:

$$S[A_{\Omega}^{m},\tau,a] : N \to A_{\Omega}^{m}$$

$$S[A_{\Omega}^{m},\tau,a](o) = a$$

$$S[A_{\Omega}^{m},\tau,a](n) = \tau(S[A_{\Omega}^{m},\tau,a](n-1)) \qquad \text{for } n>0.$$

Let τ be a derived transformation on A_{Ω}^{m}, then the following set of transformations is called a splinter of transformations on A_{Ω}^{m}.

$\{\Pi_m,\tau,\tau^2,\ldots\}$, where τ^n is defined as follows:

$$\tau^n : A^m \to A^m$$

$$\tau^n(a) = S[A_{\Omega}^{n},\tau,a](n) .$$

Because of the associativity of composition and the identity action of the projection operations, we obtain the following:

<u>Theorem 1.2.16</u>

Given an Ω-algebra A_{Ω} and let $S = \{\Pi_m,\tau,\tau^2,\ldots\}$ be a splinter of transformation on A_{Ω}^{m}, then S with the operation composition forms a cyclic monoid generated by τ.

<u>Proof</u>:

Clearly, from what we said $\tau^{a+b} = \tau^a \circ \tau^b$ and

$\Pi_m \circ \tau^a = \tau^a \circ \Pi_m = \tau^a$

For consistency we have the following:

Definition 1.2.17

For any splinter transformation τ on A_Ω^m

$$\Pi_m = \tau^0$$

Definition 1.2.18

Let $W_\Omega(X)$ be an Ω-word algebra, where $X = \{x_1,\ldots,x_n\}$ and let $y = (y_1,y_2,\ldots,y_n)$ be an n-tuple of Ω-words in $W_\Omega(X)$, let

$\tau = (\phi(y_1), \phi(y_2),\ldots,\phi(y_n))$ be the corresponding n-tuple of

n-ary operations, where for each i, $1\le i\le n$, $\phi(y_i)$ is given

$$\phi(y_i) : (X \cup \Omega)^{*n} \to (X \cup \Omega)^*,$$

as defined by

Definition 1.2.6, then

τ is called an n-ary Ω-word transformation, and the set $\{\tau^0,\tau^1,\tau^2,\ldots\}$ is called an n-ary Ω-word transformation splinter.

Definition 1.2.19

Let $W_\Omega(X)$ be an Ω-word algebra with the finite ordered set of generators $X = (x_1,\ldots,x_n)$, let τ be an n-ary Ω-word transformation, let x be an Ω-word, then the triple $H_\Omega=(W_\Omega(X),\tau,x)$ is called an OL Ω-word system.

$L[H_\Omega]$, the language of the Ω-OL word system, is defined as the following set:

$$L[H_\Omega] = \{x(\tau^m(X)) \mid m \geq 0\}$$

$WS[H_\Omega]$, the Ω-word sequence defined by H_Ω is the following mapping:

$$WS[H_\Omega] : N \to W_\Omega(X)$$

$$WS[H_\Omega](m) = x(\tau^m(X)) \quad .$$

Theorem 1.2.20

Given an OL Ω-word system $H_\Omega = (W_\Omega(X), \tau, x)$, where X is any (non-empty) finite ordered alphabet, then for any non-negative integers such that $i+j = m$, the following equality holds:

$$WS[H_\Omega](m) = x(\tau^i(X)(\tau^j(X))) \quad .$$

Proof:

Let n be the cardinality of X, then for any n-tuple $a \in (X \cup \Omega)^{*n}$ and for any n-ary transformation χ, $\chi(X)(a) = \chi(a)$, therefore

$$\tau^i(X)(\tau^j(X)) = \tau^i(\tau^j(X)) = \tau^{i+j}(X) = \tau^m(X)$$

hence

$$x(\tau^i(X)(\tau^j(X))) = x(\tau^m(X)) = S[H_\Omega](m) \quad .$$

Corollary 1.2.20

$$WS[H_\Omega](m) = x(\tau^{m-1}(X)(\tau(X))$$

$$WS[H_\Omega](m) = x(\tau(X)(\tau^{m-1}(X)) \qquad \text{if } m>0.$$

Corollary 1.2.21

Let $X = (x_1,x_2,\ldots,x_n)$, and let $H_{k\Omega} = (W_\Omega(X),\tau,x_k)$ for all k, $1\leq k\leq n$, let

$V_i = (WS[H_{1\Omega}](j), WS[H_{2\Omega}](j),\ldots,WS[H_{n\Omega}](j))$

then

$WS[H_\Omega](m) = x(\tau^{i}(V_j))$ for each i,j and m such that $i+j=n$.

In particular,

$$WS[H_\Omega](m) = x(\tau(V_{m-1})) = x(\tau^{m-1}(V_1))$$

if $m>0$.

Proof:

$$V_j = \tau^j(X).$$

Example 1.2.22

Let $X = (a,b)$, $\Omega = \{0\}$, $a(o) = 2$,

$\tau(\alpha,\beta) = (\Pi_b(\alpha,\beta) , o(\Pi_b(\alpha,\beta) , \Pi_a(\alpha,\beta)))$

where

$\Pi_a(\alpha,\beta) = \alpha$ and $\Pi_b(\alpha,\beta) = \beta$ for all α,β in $(X \cup \Omega)^*$

Let $H = (W_\Omega(X),\tau,x)$, where $x = abo$.

From this we obtain the following:

$$\tau^2(\alpha,\beta) = (o(\Pi_b(\alpha,\beta),\Pi_a(\alpha,\beta)),o(o(\Pi_b(\alpha,\beta),\Pi_a(\alpha,\beta)),\Pi_b(\alpha,\beta)))$$

$$\tau^3(\alpha,\beta) = (o(o(\Pi_b(\alpha,\beta),\Pi_a(\alpha,\beta)),\Pi_b(\alpha,\beta)),$$
$$o(o(o(\Pi_b(\alpha,\beta),\Pi_a(\alpha,\beta)),\Pi_b(\alpha,\beta),o(\Pi_b(\alpha,\beta),\Pi_a(\alpha,\beta))))$$

$$\tau(X) = (b,o(b,a)) = (b,bao)$$

$$\tau^2(X) = (o(b,a),o(o(b,a),b)) = (bao,o(bao,b)) = (bao,baobo)$$

$$\tau^3(X) = (o(o(b,a),b),o(o(o(b,a),b),o(b,a))) =$$
$$= (o(bao,b),o(o(bao,b),bao)) =$$
$$= (baobo,o(baobo,bao)) = (baobo,baobobaoo)$$

$$\tau^2(\tau(X)) = \tau^2(b,bao) = (baobo,baobobaoo)$$

$$\tau(\tau^2(X)) = \tau(bao,baobo) = (baobo,baobobaoo)$$

$WS[H](3) = x(baobo,baobobaoo) = baobobaobobaooo$

$x(\tau(X)(\tau^2(X))) = abo((b,bao)((bao,baobo))) =$

$= abo(b(bao,baobo),bao(bao,baobo))$

$= abo(baobo,baobobaoo) =$

$= baobobaobobaooo$

$x(\tau^2(X)(\tau(X))) = abo((bao,baobo)((b,bao))) =$

$= abo((bao(b,bao),baobo(b,bao))) =$

$= abo(baobo,baobobaoo) =$

$= baobobaobobaooo.$

The image of an OL Ω-word system under a homomorphism h, is called Ω-OL system. It is clear that a statement similar to Theorem 1.2.20 may be phrased about the image, since we may commute h and τ. One way we may interpret Corollary 1.2.20 is as follows:

In a developmental system without interaction, the global pattern on the highest level is identical with the pattern of the first stage of the development, or in cosmic terms: The macro-cosmos reflects the microcosmos.