Non-planar Core Reduction of Graphs

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Abstract. We present a reduction method that reduces a graph to a smaller core graph which behaves invariant with respect to planarity measures like crossing number, skewness, and thickness. The core reduction is based on the decomposition of a graph into its triconnected components and can be computed in linear time. It has applications in heuristic and exact optimization algorithms for the planarity measures mentioned above. Experimental results show that this strategy yields a reduction to 2/3 in average for a widely used benchmark set of graphs.

1 Introduction

Graph drawing is concerned with the problem of rendering a given graph on the two-dimensional plane so that the resulting drawing is as readable as possible. Objective criteria for the readability of a drawing depend mostly on the application domain, but achieving a drawing without edge crossings is in general a primary objective. Such a drawing is called a *planar* drawing. However, it is well known that not every graph can be drawn without edge crossings. The famous theorem by Kuratowski [10] shows that a graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

If a graph G is not planar, a question arises naturally: How far away is the graph from planarity? For that reason, various measures for non-planarity have been proposed. The most prominent one is the *crossing number* of a graph which asks for the minimum number of crossings in any drawing of G. Further measures are the *skewness* which is the minimum number of edges we have to remove from G in order to obtain a planar graph, and the *thickness* which is the minimum number of planar subgraphs of G whose union is G. However, finding an optimal drawing with respect to any of these non-planarity measures yields an NP-hard optimization problem [5, 12, 13].

Various heuristic and exact methods for solving these optimization problems have been proposed; please refer to [11, 14, 7] for an overview. It is well known that it is sufficient to consider each biconnected component of the graph separately. We present a new approach based on the triconnectivity structure of the graph which reduces a 2-connected graph to a core that behaves invariant to the above non-planarity measures. We call this core graph the non-planar core $\mathcal{C}$ of G and show that it can be constructed in linear computation time. In order to compute the crossing number, skewness, or thickness of G, any standard algorithm can be applied to $\mathcal{C}$. This approach targets in particular exact

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algorithms, since their running times heavily depend on the instance size. It is also constructive in the sense that we can reconstruct a solution for G (e.g., a crossing minimal drawing) from the solution for the core graph $\mathcal{C}$.

This paper is organized as follows. After introducing some basic terminology, the non-planar core is defined in Sect. 3. The next three sections 4–6 apply the new reduction technique to crossing number, skewness and thickness. Section 7 shows that a straight-forward idea to further reduce the size of the core is not possible. We conclude the paper with experimental results.

2 Preliminaries

Let $G = (V, E)$ be a graph. If $(u, v) \in V \times V$, we use $G \cup (u, v)$ as a shorthand for the graph $(V, E \cup (u, v))$. For a subset of the vertices $V' \subseteq V$, we denote with $G[V']$ the vertex induced subgraph (V', E_V), where $E_V \subseteq E$ is the set of edges with both end vertices in V'. If $E' \subseteq E$ is a subset of the edges of G, we denote with $G[E']$ the subgraph induced by the edges in E', that is $G[E'] = (V_E, E')$ with $V_E = \{v \in V \mid v \text{ is incident with an edge in } E'\}$. Suppose that G is planar and let Γ be an embedding of G with face set F. The *dual graph* $\Gamma^* = (F, E^*)$ of Γ contains an edge $e^* = (f, f')$ for every edge $e \in E$ such that e is on the boundary of both f and f'; edge e is also called the *primal edge* of e^*.

2.1 Crossing Number, Skewness, and Thickness

The *crossing number* $\nu(G)$ of a graph $G = (V, E)$ is the minimum number of crossings in any drawing of G. The *skewness* $\mu(G)$ of G is the size of a minimum cardinality edge set F such that $G[E \setminus F]$ is planar, and we call $G[E \setminus F]$ a maximum planar subgraph of G. The *thickness* $\theta(G)$ of G is the minimum number k of planar graphs $G_1, \ldots, G_k$ such that $G_1 \cup \ldots \cup G_k = G$.

We extend the notion of crossing number and skewness to graphs with a given weight function $w : E \to \mathbb{N}$. We call the sum

$$\sum_{\substack{e,f \in E \\ e \text{ crosses } f}} w(e) \cdot w(f)$$

the *crossing weight* of a drawing, and we denote with $\nu(G, w)$ the *weighted crossing number* of G which is the minimum crossing weight of any drawing of G. If $E' \subseteq E$, we define $w(E') := \sum_{e \in E'} w(e)$ to be the weight of E', and we denote with $\mu(G, w)$ the *weighted skewness* of G which is the weight $w(F)$ of a minimum weight edge set F such that $G[E \setminus F]$ is planar.

In the remainder of this paper, we will restrict our attention to 2-connected graphs. However, the results on crossing number, skewness, and thickness can easily be generalized using the following relationships. Let G be a graph and $B_1, \ldots, B_k$ its biconnected components. Then,

$$\nu(G) = \sum_{i=1,\ldots,k} \nu(B_i), \quad \mu(G) = \sum_{i=1,\ldots,k} \mu(B_i), \quad \theta(G) = \max_{i=1,\ldots,k} \theta(B_i).$$

2.2 Minimum Cuts and Traversing Costs

A *cut* in G is a partition $(S, \bar{S})$ of the vertices of G. The *capacity* $c(S, \bar{S})$ of the cut is the cardinality of the set $E(S, \bar{S})$ of all the edges connecting vertices in S with vertices in $\bar{S}$. For two vertices $s, t \in V$, we call $(S, \bar{S})$ an st-cut if s and t are in different sets of the cut. A *minimum st-cut* is an st-cut of minimum capacity. We denote the capacity of a minimum st-cut in G with $\text{mincut}_{s,t}(G)$.

Let $s, t \in V$ and $G \cup (s, t)$ be 2-connected and planar. For an embedding Γ of $G \cup (s, t)$, we define the *traversing costs* of Γ with respect to (s, t) to be the shortest path in the dual graph of Γ that connects the two faces adjacent to (s, t) without using the dual edge of (s, t). We also call the corresponding list of primal edges a *traversing path* for s and t. Gutwenger, Mutzel, and Weiskircher [8] showed that the traversing costs are independent of the choice of the embedding Γ of G. Hence, we define the *traversing costs* of G with respect to (s, t) to be the traversing costs of an arbitrary embedding Γ with respect to (s, t). It is easy to see that a traversing path defines an st-cut. The following theorem shows that this st-cut is even a minimum st-cut.

Theorem 1. *Let $G = (V, E)$ be a graph with $s, t \in V$ and $G \cup (s, t)$ is 2-connected and planar. Then, the traversing costs of G with respect to (s, t) are equal to* $\text{mincut}_{s,t}(G)$.

We are interested in special subgraphs of a 2-connected, not necessarily planar graph $G = (V, E)$ which we call planar st-components. Let $s, t \in V$ be two distinct vertices. We call an edge induced subgraph $C = G[E_C]$ a *planar st-component* of G if $G \cup (s, t)$ is 2-connected and planar, and if $V(C) \cap V' \subseteq \{s, t\}$, where $V' := V(G[E \setminus E_C])$ denotes the vertex set of the graph induced by the edges not contained in C. Obviously, since G is 2-connected, $V(C) \cap V'$ is either empty or contains both s and t.

2.3 SPQR-Trees

SPQR-trees basically represent the decomposition of a biconnected graph into its triconnected components. For a formal definition we refer the reader to [4, 3]. Informally speaking, the nodes of an SPQR-tree $\mathcal{T}$ of a graph G stand for serial (S-nodes), parallel (P-nodes), and triconnected (R-nodes) structures, as well as edges of G (Q-nodes). The respective structure is given by skeleton graphs associated with each node of $\mathcal{T}$, which are either cycles, bundles of parallel edges, or triconnected simple graphs. We denote with $skeleton(\eta)$ the skeleton graph associated with node η. Each edge $e \in skeleton(\eta)$ corresponds to a tree edge $e_{\mathcal{T}} = (\eta, \xi)$ incident with η. We call ξ the *pertinent node* of e. The edge e stands for a subgraph called the *expansion graph* of e that is only attached to the rest of the graph at the two end vertices of e. The expansion graph of e is obtained as follows. Deleting edge $e_{\mathcal{T}}$ splits $\mathcal{T}$ into two connected components. Let $\mathcal{T}_\xi$ be the connected component containing ξ. The *expansion graph* of e (denoted with $\text{expansion}(e)$) is the graph induced by the edges that are represented by the Q-nodes in $\mathcal{T}_\xi$. We further introduce the notation $\text{expansion}^+(e)$ for the graph $\text{expansion}(e) \cup e$.

For our convenience, we omit Q-nodes and distinguish in skeleton graphs between *real edges* that are skeleton edges whose pertinent node would be a Q-node, and *virtual edges.*

3 The Non-planar Core

Let G be a 2-connected graph and let $\mathcal{T}$ be its SPQR-tree. For a subtree $\mathcal{S}$ of $\mathcal{T}$, we define the *induced graph* $G[\mathcal{S}]$ of $\mathcal{S}$ to be the edge induced subgraph $G[E']$, where E' is the union of all edges in skeletons of nodes of $\mathcal{S}$ that have no corresponding tree edge in $\mathcal{S}$:

$$E' := \bigcup_{\eta \in \mathcal{S}} \{e \in \mathit{skeleton}(\eta) \mid e \text{ has no corresponding tree edge in } \mathcal{S}\}$$

Hence, the induced graph consists of virtual edges representing planar *st*-components and real edges representing edges of G. Analogously to SPQR-trees, we define the expansion graph of a virtual edge in $G[\mathcal{S}]$ and use the notations expansion(e) and expansion$^+$(e) for a virtual edge e. We can reconstruct G from $G[\mathcal{S}]$ by replacing every virtual edge with its expansion graph. We have in particular $G[\mathcal{T}] = G$.

We define the *non-planar core* of G to be the empty graph if G is planar, and the induced graph of the smallest non-empty subtree $\mathcal{S}$ of $\mathcal{T}$ such that the *expansion*$^+(e)$ is planar for every virtual edge e in $G[\mathcal{S}]$. It is easy to derive the following properties of the non-planar core of G.

Lemma 1. *Let $\mathcal{C} = G[\mathcal{S}]$ be the non-planar core of G.*

(a) $\mathcal{C} = \emptyset \iff G$ is planar
(b) $\mathcal{C} \neq \emptyset \implies$ Every leaf of $\mathcal{S}$ is an R-node with non-planar skeleton.

Proof. The first part follows directly from the definition.

Let $\mathcal{C} \neq \emptyset$ and thus G be non-planar. Then, $\mathcal{S}$ must contain a node with non-planar skeleton. Suppose $\xi \in \mathcal{S}$ is a leaf whose skeleton is planar. Since $\mathcal{S}$ contains at least one further node, ξ has exactly one adjacent node η in $\mathcal{S}$. But then the expansion graph of the virtual edge of ξ in $\mathit{skeleton}(\eta)$ is planar, and hence $\mathcal{S}' := \mathcal{S} - \xi$ is also a subtree of $\mathcal{T}$ with the property that expansion$^+(e)$ is planar for every virtual edge e in $G[\mathcal{S}']$. This is a contradiction to the minimality of $\mathcal{S}$. It follows that every leaf of $\mathcal{S}$ is a node with non-planar skeleton. This must be an R-node, since only R-node skeletons can be non-planar. □

We extend the non-planar core $\mathcal{C}$ of G by an additional weight function $w : E(\mathcal{C}) \to \mathbb{N}$. If e is a real edge, then $w(e)$ is 1. Otherwise, let $e = (s,t)$ and we define $w(e) := \mathrm{mincut}_{s,t}(\mathrm{expansion}(e))$. We denote the non-planar core with given edge weights by a pair $(\mathcal{C}, w)$.

Theorem 2. *Let $G = (V,E)$ be a 2-connected graph. Then, the non-planar core of G and the corresponding edge weights can be computed in $\mathcal{O}(|V| + |E|)$ time.*

Algorithm 1. Computation of the non-planar core.

Require: 2-connected graph $G = (V, E)$
Ensure: non-planar core $(\mathcal{C}, w)$ of G

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Let 𝒯 be the (undirected) SPQR-tree of G

Let candidates be an empty stack of nodes
for all ξ ∈ 𝒯 do
   d[ξ] := deg(ξ)
   if d[ξ] = 1 then
      candidates.push(ξ)
   end if
end for

P := ∅
while candidates ≠ ∅ do
   ξ := candidates.pop()
   if skeleton(ξ) is planar then
      P := P ∪ {ξ}
      for all η ∈ Adj(ξ) do
         d[η] := d[η] − 1
         if d[η] = 1 then
            candidates.push(η)
         end if
      end for
   end if
end while

Let 𝒮 be the graph induced by the vertices in V(𝒯) \ P
𝒞 := G[𝒮]

for all edges e ∈ 𝒞 do
   if e is a virtual edge then
      w(e) := traversing costs of expansion(e) with respect to e
   else
      w(e) := 1
   end if
end for
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Proof. Algorithm 1 shows a procedure for computing the non-planar core. We achieve linear running time, since constructing an SPQR-tree, testing planarity, and computing traversing costs takes only linear time; see [6, 9, 8].

4 Crossing Number

In this section, we apply the non-planar core reduction to the crossing number problem. The following theorem shows that it is sufficient to compute the crossing number of the non-planar core.

Theorem 3. *Let G be a 2-connected graph, and let $(\mathcal{C}, w)$ be its non-planar core. Then,*

$$\nu(G) = \nu(\mathcal{C}, w).$$

The proof of Theorem 3 is based on the following lemma which allows us to restrict the crossings in which the edges of a planar st-component may be involved so that we can still obtain a crossing minimal drawing of G. A similar result has been reported by Širáň in [15]. However, as pointed out in [1], the proof given by Širáň is not correct.

Lemma 2. *Let $C = (V_C, E_C)$ be a planar st-component of $G = (V, E)$. Then, there exists a crossing minimal drawing $\mathcal{D}^*$ of G such that the induced drawing $\mathcal{D}^*_C$ of C has the following properties:*

(a) *$\mathcal{D}^*_C$ contains no crossings;*
(b) *s and t lie in a common face f_{st} of $\mathcal{D}^*_C$;*
(c) *all vertices in $V \setminus V_C$ are drawn in the region of $\mathcal{D}^*$ defined by f_{st};*
(d) *there is a set $E_s \subseteq E_C$ with $|E_s| = \mathrm{mincut}_{s,t}(C)$ such that any edge $e \in E \setminus E_C$ may only cross through all edges of E_s, or through none of E_C.*

Proof. Let $G' = G[E \setminus E_C]$ be the graph that results from cutting C out of G. Let $\mathcal{D}$ be an arbitrary, crossing minimal drawing of G, and let $\mathcal{D}_C$ (resp. $\mathcal{D}'$) be the induced drawing of C (resp. G'). We denote by P the planarized representation of G' induced by $\mathcal{D}'$, i.e. the planar graph obtained from $\mathcal{D}'$ by replacing edge crossings with dummy vertices. Let Γ_P be the corresponding embedding of P and Γ^*_P the dual graph of Γ_P.

Let $p = f_1, \ldots, f_{k+1}$ be a shortest path in Γ^*_P that connects an adjacent face of s with an adjacent face of t. There are $\lambda := \mathrm{mincut}_{s,t}(C)$ edge disjoint paths from s to t in C. Each of these λ paths crosses at least k edges of G' in the drawing $\mathcal{D}$. Hence, there are at least $\lambda \cdot k$ crossings between edges in C and edges in G'. We denote with E_p the set of primal edges of the edges on the path p. Let $\mathcal{D}^*_C$ be a planar drawing of C in which s and t lie in the same face f_{st}, and let E_s be the edges in a traversing path in $\mathcal{D}^*_C$ with respect to s and t. By Theorem 1, there is a minimum st-cut $(S, \bar{S})$ with $E(S, \bar{S}) = E_s$, and thus $|E_s| = \lambda$. We can combine $\mathcal{D}'$ and $\mathcal{D}^*_C$ by placing the drawing of $C[S]$ in face f_1 and the drawing of $C[\bar{S}]$ in f_{k+1}, such that all the edges in E_p cross all the edges in E_s; see Fig. 1. It is easy to verify that the conditions (a)–(d) hold for the resulting drawing $\mathcal{D}^*$. $\square$

We conclude this section with the proof of Theorem 3, i.e. we show that $\nu(G) = \nu(\mathcal{C}, w)$.

Proof (of Theorem 3).

"$\leq$" Let $\mathcal{D}_\mathcal{C}$ be a drawing of $\mathcal{C}$ with minimum crossing weight. For each virtual edge $e = (s,t) \in \mathcal{C}$, we replace e by a planar drawing $\mathcal{D}_e$ of the corresponding planar st-component so that all edges that cross e in $\mathcal{D}_\mathcal{C}$ cross the edges in a traversing path in $\mathcal{D}_e$ with respect to (s,t). Since w_e is equal to the

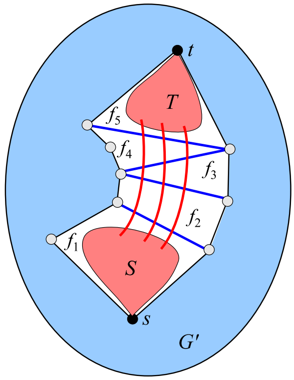


Fig. 1. Final drawing $\mathcal{D}^*$ of G; here, $p = f_1, f_2, f_3, f_4, f_5$ is the shortest path in Γ_P^*

traversing costs of $\mathcal{D}_e$ with respect to (s,t) by definition, replacing all virtual edges in this way leads to a drawing of G with $\nu(\mathcal{C}, w)$ crossings, and hence $\nu(G) \leq \nu(\mathcal{C}, w)$.

"$\geq$" On the other hand, let $\mathcal{D}$ be a crossing minimal drawing of G. For each virtual edge $e = (s,t) \in \mathcal{C}$, we modify $\mathcal{D}$ in the following way. Let C be the planar st-component corresponding to e, and let G' be the rest of the graph. By Lemma 2, we obtain another crossing minimal drawing of G if we replace the drawing of C with a planar drawing $\mathcal{D}_C$ of C such that all edges of G' that cross edges in C will cross the edges in $E(S, \bar{S})$, where $(S, \bar{S})$ is a minimum st-cut in C. If we replace $\mathcal{D}_C$ with an edge $e = (s,t)$ with weight $w(e) := |E(S, \bar{S})| = \mathrm{mincut}_{s,t}(C)$, we obtain a drawing with the same crossing weight.

By replacing all virtual edges in that way, we obtain a drawing of $\mathcal{C}$ whose crossing weight is the crossing number of G. It follows that $\nu(G) \geq \nu(\mathcal{C}, w)$, and hence the theorem holds. □

5 Skewness

We can apply the non-planar core reduction to the skewness of a graph in a rather analogue way. The following lemma establishes our main argument.

Lemma 3. *Let $G = (V, E)$ be a 2-connected graph, $C = (V_C, E_C)$ a planar st-component of G, and $P = (V, E_P)$ a maximum planar subgraph of G. Then, either $C \subseteq P$, or $|E_C| - |E_P \cap E_C| = \mathrm{mincut}_{s,t}(C)$.*

Proof. We distinguish two cases.

Case 1. *There is a path from s to t in P which consists only of edges of C.* Consider an embedding Γ of P. If we cut out C from Γ, then s and t must lie in a common face of the resulting embedding Γ'. On the other other hand, we can construct an embedding Γ_C of C in which s and t lie on the external face. Inserting Γ_C into Γ' yields an embedding of $P \cup C$. Since P is a maximum planar subgraph of G and $C \subseteq G$, it follows that $C \subseteq P$.

Case 2. *There is no such path from s to t.* Let $E' = E_P \cap E_C$ be the edges of C contained in P. It follows that $C' = (V_C, E')$ has at least two connected components, one containing s, and the other containing t. Hence, the number of edges in $E_C \setminus E'$ is at least $\mathrm{mincut}_{s,t}(C)$, which implies $|E_C| - |E_P \cap E_C| \geq \mathrm{mincut}_{s,t}(C)$.

On the other hand, we can construct an embedding of C with s and t on the external face, and remove the $\mathrm{mincut}_{s,t}(C)$ edges in a traversing path of C with respect to (s,t). This yields an embedding Γ with two connected components C_s and C_t with $s \in C_s$ and $t \in C_t$. Let $G' = G[E \setminus E_C]$ be the rest of the graph. Since C_s has only s in common with G' and C_t has only t in common with G', we can insert Γ into any embedding of $G' \cap P$ preserving planarity. This implies that $|E_C| - |E_P \cap E_C| \leq \mathrm{mincut}_{s,t}(C)$ and the lemma holds. □

Using this lemma, we can show that the non-planar core is invariant with respect to skewness.

Theorem 4. *Let G be a 2-connected graph, and let $(\mathcal{C}, w)$ be its non-planar core. Then,*

$$\mu(G) = \mu(\mathcal{C}, w)$$

Proof. Let $G = (V, E)$ and $\mathcal{C} = (V_{\mathcal{C}}, E_{\mathcal{C}})$.

"≥" Let $P = (V, E_P)$ be a maximum planar subgraph of G. We have $\mu(G) = |E| - |E_P|$. We show that we can construct a planar subgraph $P_{\mathcal{C}} = (V_{\mathcal{C}}, E')$ of $\mathcal{C}$ with $w(E_{\mathcal{C}}) - w(E') = \mu(G)$.

Consider a planar st-component C of G. By Lemma 3, we know that either C is completely contained in P, or exactly $\mathrm{mincut}_{s,t}(C)$ many edges of C are not in E_P. In the first case, we know that an st-path is in P, and hence replacing C by the corresponding edge (s,t) preserves planarity. In the second case, the corresponding virtual edge $e = (s,t)$ with weight $w(e) = \mathrm{mincut}_{s,t}(C)$ will not be in $P_{\mathcal{C}}$.

Constructing $P_{\mathcal{C}}$ in this way obviously yields a planar subgraph $(V_{\mathcal{C}}, E')$ of $\mathcal{C}$ with $w(E_{\mathcal{C}} \setminus E') = \mu(G)$.

"≤" Let $P_{\mathcal{C}} = (V_{\mathcal{C}}, E_P)$ be a maximum weight planar subgraph of $\mathcal{C}$, and let D be a drawing of $P_{\mathcal{C}}$. We have $\mu(\mathcal{C}, w) = w(E_{\mathcal{C}}) - w(E_P)$. We show that we can construct a planar subgraph $P' = (V, E')$ of G with $|E| - |E'| = \mu(\mathcal{C}, w)$. We again consider a planar st-component C of G. Let $e = (s,t)$ be the corresponding virtual edge, and let D_C be a planar drawing of C in which both s and t lie in the external face. If e is in $P_{\mathcal{C}}$, we can replace e with the drawing D_C and the resulting drawing remains planar. If e is not in $P_{\mathcal{C}}$, we remove the edges of a traversing path of C with respect to (s,t) from D_C. This yields a drawing D'_C with two connected components, one containing s, and the other containing t. Obviously, we can add the drawing D'_C to D preserving planarity, and we removed exactly $w(e) = \mathrm{mincut}_{s,t}(C)$ edges from G.

We finally end up with a drawing of a planar subgraph $P = (V, E')$ of G with $|E| - |E'| = \mu(\mathcal{C}, w)$. □

6 Thickness

For computing the thickness of G, we do not need to consider the weight of edges in the non-planar core $\mathcal{C}$ of G. Instead, we slightly modify $\mathcal{C}$ by splitting every virtual edge (s,t) whose expansion graph does not contain an edge (s,t). We denote the resulting graph with $\mathrm{core}^+(G)$.

Theorem 5. *Let G be a 2-connected graph, and let $\mathcal{C}' = \mathrm{core}^+(G)$. Then,*

$$\theta(G) = \theta(\mathrm{core}^+(G))$$

Proof. "$\geq$" Let $\theta(G) = k$, and let $G_1, \ldots, G_k$ be k planar graphs with $G_1 \cup \ldots \cup G_k = G$. We consider a planar st-component C. We distinguish two cases:

(i) If there is a graph G_i such that $G_i \cap C$ contains a path from s to t, then we remove all edges and vertices $\neq s,t$ of C from all graphs $G_1, \ldots, G_k$, and we add the edge $e = (s,t)$ to G_i. If C does not contain an edge (s,t), then we also split e.

(ii) Otherwise, we know that $k \geq 2$, and therefore there are two graphs G_i and G_j with $i \neq j$. We add the edges $e_s = (s,d)$ to G_i and $e_t = (d,t)$ to G_j, where d is a new dummy vertex. If any of the end vertices of e_1 (resp. e_2) is not yet contained in G_i (resp. G_j), we also add this vertex.

It follows that we can construct k planar graphs whose union is $\mathrm{core}^+(G)$, and thus $\theta(G) \geq \theta(\mathrm{core}^+(G))$.

"$\leq$" Let $\theta(\mathrm{core}^+(G)) = k$, and let $G_1, \ldots, G_k$ be k pairwise edge disjoint planar graphs with $G_1 \cup \ldots \cup G_k = \mathrm{core}^+(G)$. We consider a virtual edge $e = (s,t)$ of the non-planar core of G. Let $C = (V_C, E_C)$ be the expansion graph of e. If C contains an edge (s,t), then e is contained in $\mathrm{core}^+(G)$, and thus there is a subgraph, say G_i, containing e. We replace e in G_i by C.

Otherwise, C contains an edge $e = (s,t)$ and e was split into two edges, say $e_1 = (s,d)$ and $e_2 = (d,t)$, in $\mathrm{core}^+(G)$. We split C into two edge disjoint graphs C_1 and C_2 in the following way: Let E' be the set of edges incident with s. Then, C_1 is the graph induced by E', and C_2 is the graph induced by $E_C \setminus E'$. Let G_i be the graph containing e_1, and let G_j be the graph containing e_2. If $i = j$, then we replace e_1 and e_2 by C in G_i. Otherwise, we replace e_1 by C_1 in G_i, and e_2 by C_2 in G_j.

It follows that we can construct k planar subgraphs of G whose union is G, and thus $\theta(G) \leq k$. □

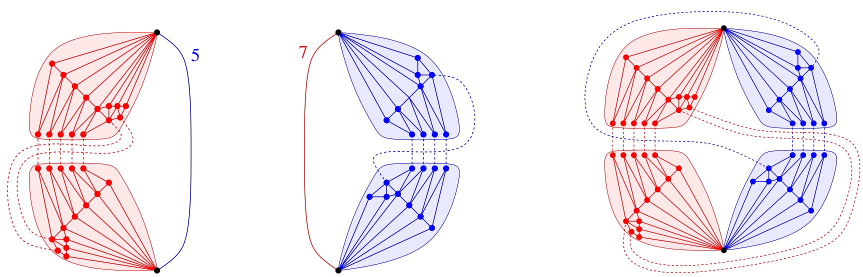
7 Further Reductions

It is a straight-forward idea to try to reduce the computation of crossing number or skewness to the non-planar skeletons of R-nodes. To do this, it would be necessary to be able to merge two components with the following properties:

(a) Both components have exactly two nodes, say s and t, in common.
(b) Each component is – if augmented with a virtual edge (s, t) – non-planar and at least 2-connected.
(c) The crossing number (skewness) of the merged component is the sum of the crossing numbers (skewnesses) of the components.

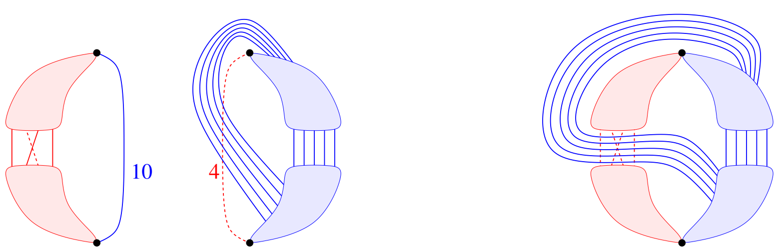
In the following we will give counterexamples to show that this approach fails.

Crossing Number. Figure 2(a) shows two components and their crossing minimal embedding, with regards to the minimum st-cut of their counterpart, which defines the weight of the virtual edges. The two components have unique minimum st-cuts, denoted by dashed lines. The minimum st-cut of the left component is 7, whereby the minimum st-cut of the right one is 5. The minimum crossing numbers of the left and right components are 10 and 4, respectively; but the minimum crossing number of the merged result is only $2 \cdot 4 + 5 = 13$ (Fig. 2(b)), which is less than the sum $10 + 4 = 14$. The reason is that we have edges that partially cross through the counterpart component.



(a) Crossing minimal (comp.-wise): 10 and 4 (b) Crossing minimal: 13

Fig. 2. Calculating only the crossing numbers of the non-planar R-nodes is not correct



(a) Skewness (component-wise): 1 and 4 (b) Skewness (as a whole): 4

Fig. 3. Calculating only the skewnesses of the non-planar R-nodes is not correct

Skewness. Figure 3(a) shows two components including the virtual edges with the weights of their counterpart's minimum st-cut. The jelly bag cap shaped regions denote dense, crossing-free, 3-connected subgraphs, similar to the ones in Fig. 2. The edges which have to be removed to get a planar subgraph are the dashed lines. The skewness of the left component is 1 — note that the choice between the two possibilities is arbitrary. The skewness of the right component corresponds to removing its virtual edge, and therefore has the value of 4. We can see that we have one edge that has to be removed for both components, and is therefore counted twice: the merged drawing has a skewness of only 4, although the sum of the separate skewnesses would have suggested $1 + 4 = 5$. Note that we can not even find any set of edges which does not include the virtual edge, has the size 5, and can be removed in order to get a planar subgraph.

8 Experimental Results and Discussion

We tested the effect of our reduction strategy on a widely used benchmark set commonly known as the *Rome library* [2]. This library contains over 11.000 graphs ranging from 10 to 100 vertices, which have been generated from a core set of 112 graphs used in real-life software engineering and database applications.

We found that all non-planar graphs in the library have a single non-planar biconnected component whose non-planar core is the skeleton of just one R-node. Fig. 4 shows the average relative size of the non-planar core $\mathcal{C}$ compared to the non-planar biconnected component (block) and the total graph. Here, the size of a graph is simply the number of its edges. It turns out that, on average, the size of the non-planar core is only 2/3 of the size of the non-planar block. Compared to the whole graph, the size of the non-planar core reduces to about 55% on average. This shows that the new approach provides a significant improvement for reducing the size of the graph.

It will be interesting to see the effect the reduction strategy has on the practical performance of heuristics and exact algorithms for computing crossing number, skewness, and thickness. It remains an open problem if we can further reduce

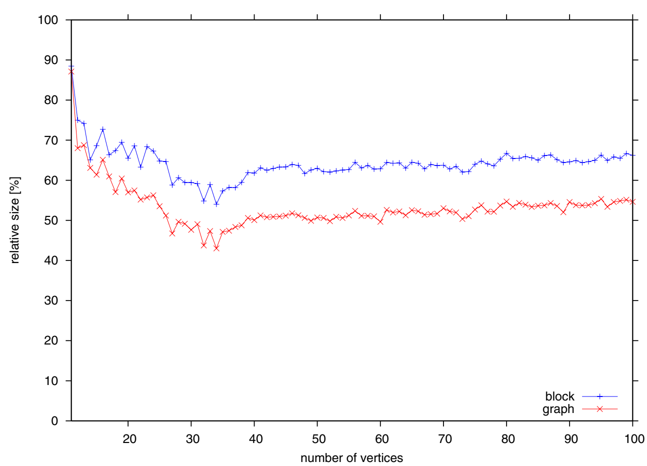


Fig. 4. Relative size of the non-planar core for the Rome graphs

a graph based on its connectivity structure. In particular, there might be the possibility for improvements by considering cut sets with three or more vertices.

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