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ERRATUM

Erratum to: Extrinsic diameter of immersed flat tori in S^3

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Abstract In the paper referred to in the title, we proved that any immersed flat tori in S^3 whose mean curvature does not change sign has extrinsic diameter π. Although the main result there is correct, there is a gap of the proof of this fact. The purpose here is to correct the previous paper's argument and clarify the statement.

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Introduction

Let S^3 be the unit sphere in $\mathbf{R}^4 = \mathbf{C}^2$. The Clifford torus in S^3 given by

$$M_\theta := \left\{ (z, w) \in C^2 \,;\, |z|^2 = \cos^2\theta, \quad |w|^2 = \sin^2\theta \right\}, \quad 0 < \theta < \pi/2,$$

is a flat embedded torus of constant mean curvature. We denote by $\iota_\theta : M_\theta \to S^3$ the canonical inclusion, and by ds_θ^2 is the induced metric on M_θ. In the previous paper [1, Theorem 0.2], we stated the following assertion:

Theorem *Let $f : (M_\theta, ds_\theta^2) \to S^3$ be an isometric immersion whose mean curvature function does not change sign on M_θ, then f is congruent to ι_θ.*

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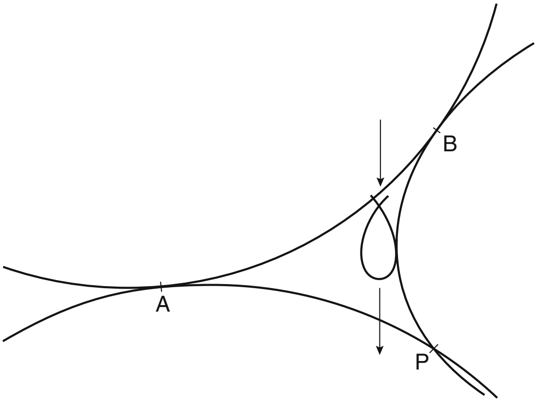

Fig. 1 A typical case where the previous argument fails

Unfortunately, there was a gap in the proof of Proposition 4.7, which was a key assertion to prove the theorem. In fact, in [1, Claim 5 (Page 133)], we stated that the subset $\mathcal{L}$ is closed in the subarc $C|_{[A,B]}$ of C. However, this is not true if s_∞ does not coincide with s_{X_∞}. In fact, $s = s_\infty$ might not be the first point where $\varphi_s(S'_2)$ meets the union of the arcs $[AP) \cup [BP)$, where $[AP)$ and $[BP)$ are subarcs of γ_1 defined in [1, Page 132]. For example, if we consider the case as in Fig. 1 , the set $\mathcal{L}$ is not closed, since the shell is tangent to $(PB]$ before attaching to $[AP)$. The purpose of this paper is to correct the proof of Proposition 4.7.

1 A key proposition

The following assertion will play a crucial role in the new proof of Proposition 4.7. (The definition of positive shell and negative shell is given in [1, Page 118].)

Proposition 1.1 *Let $\sigma : [0, a] \to S^2$ be a positive shell (respectively negative shell) whose geodesic curvature $\kappa(t)$ $(0 \leq t \leq a)$ satisfies $\kappa(t) > \kappa_0$ (respectively $\kappa(t) < -\kappa_0$), where κ_0 is a positive constant. Let Γ be the circle of geodesic curvature κ_0 $(respectively - \kappa_0)$ which is tangent to $\sigma(t)$ at $t = b$ $(0 \leq b \leq a)$. Then $\sigma([0, a]) \setminus \{\sigma(b)\}$ lies in the interior Δ_Γ of the circle Γ* (the definition of Δ_Γ is given in [1, Page 118]).

To prove this, we show the following lemma, which is a special case of the proposition.

Lemma 1.2 *Let $\sigma : [0, a] \to S^2$ be a positive shell whose geodesic curvature $\kappa(t)$ $(0 \leq t \leq a)$ satisfies $\kappa(t) > \kappa_0$, where κ_0 is a positive constant. Let Γ be the circle of geodesic curvature κ_0 which is tangent to $\sigma(t)$ at $t = 0$ or $t = a$. Then $\sigma((0, a))$ lies in the interior Δ_Γ of the circle Γ.*

Proof We consider the case $t = 0$. (The case $t = a$ is proved similarly.) Suppose that the image $\sigma((0, a))$ does not lie in Δ_Γ. The condition $\kappa(t) > \kappa_0$ yields that $\sigma(t)$ lies in Δ_Γ for sufficiently small $t > 0$. Since σ is a positive shell, $\sigma(a - t)$ also lies in Δ_Γ for sufficiently small $t > 0$. Thus, there exist two points $s_1, s_2 \in (0, a)$ satisfying the following three properties (see Fig. 2, left):

- $0 < s_1 \leq s_2 < a$,
- $\sigma(s_1)$ and $\sigma(s_2)$ lie on Γ,
- $\sigma((0, s_1))$ and $\sigma((s_2, a))$ are contained in the domain Δ_Γ.

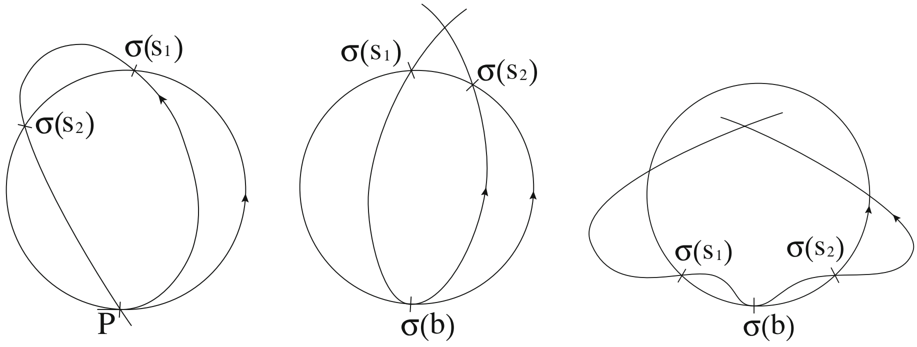

Fig. 2 Three impossible arrangements of σ

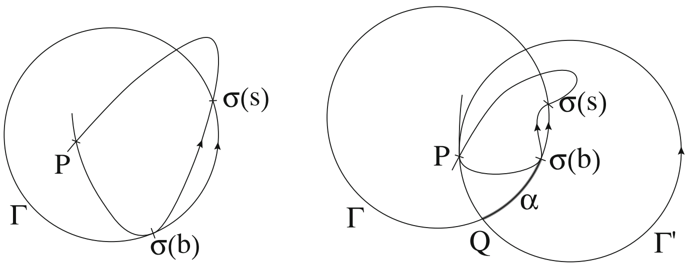

Fig. 3 The case $\sigma((0, b)) \subset \Delta_\Gamma$

Applying [1, Lemma 4.5] to the arc $\sigma([0, s_1])$, we can conclude that $\sigma(s_1)$ lies in the past part of Γ with respect to the node $P := \sigma(0) = \sigma(a)$. Since σ is a simple closed arc, $\sigma(s_2)$ must also lie in the past part of Γ with respect to P. On the other hand, applying [1, Lemma 4.5] to another arc $\sigma([s_2, a])$, we can conclude that $\sigma(s_2)$ must lie in the future part of Γ with respect to P, which is a contradiction. □

We now prove Proposition 1.1 as follows: Reversing the orientation of the curve σ if necessary, we may assume that σ is a positive shell. Since Lemma 1.2 proves the case of $b = 0$ or $b = a$, we may assume that $0 < b < a$. Suppose also that the image $\sigma([0, a]) \setminus \{\sigma(b)\}$ does not lie in Δ_Γ. The center and right of Fig. 2 are typical such cases. Suppose that both $\sigma([0, b))$ and $\sigma((b, a])$ do not lie in Δ_Γ. Then, there exist two points $s_1, s_2 \in [0, a]$ satisfying the following three properties:

- $0 \le s_1 < b < s_2 \le a$,
- $\sigma(s_1)$ and $\sigma(s_2)$ lie on Γ,
- $\sigma((s_1, b))$ and $\sigma((b, s_2))$ are contained in the domain Δ_Γ.

We can make a contradiction applying [1, Lemma 4.5] for two arcs $\sigma([s_1, b])$ and $\sigma([b, s_2])$, respectively, like as in the proof of Lemma 1.2. Thus, either $\sigma([0, b))$ or $\sigma((b, a])$ lies in Δ_Γ. In particular, P lies in Δ_Γ. We now consider the case that $\sigma([0, b))$ lies in Δ_Γ. (The case $\sigma((b, a]) \subset \Delta_\Gamma$ also makes a contradiction using the same argument.) Then, there exists a point $s \in (b, a)$ such that $\sigma(s)$ lies on Γ and $\sigma((b, s))$ lies in Δ_Γ (see Fig. 3, left). Let Γ' be the circle of geodesic curvature κ_0 which is tangent to $\sigma(t)$ at $t = 0$. Γ' intersects Γ with two points. Let Q be one of two such intersection points, which lies in the past part of Γ with respect to $\sigma(b)$. By Lemma 1.2, $\sigma((0, a))$ lies in $\Delta_{\Gamma'}$ (cf. Fig. 3, right). In particular, $\sigma(b)$

and $\sigma(s)$ lie on $\Gamma \cap \Delta_{\Gamma'}$. Then, we can conclude that $\sigma(b)$ and $\sigma(s)$ lie in the future part of Γ with respect to Q. On the other hand, applying [1, Lemma 4.5] to the arc $\sigma([b, s])$, we can conclude that $\sigma(s)$ lies in the past part of Γ with respect to $\sigma(b)$. Hence $\sigma(s)$ must lie in the subarc α of Γ bounded by Q and $\sigma(b)$ (see Fig. 3, right). However, this implies that $\sigma((b, s))$ must meet the arc $\sigma([0, b])$, which contradicts the fact that σ is a simple closed arc. This proves the proposition.

2 A modification of the definition of *r*-diamonds

In [1, Definition 4.8], we gave a definition of r-diamonds. However, we need to modify it for the new proof of Proposition 4.7. More precisely, the condition (v) of r-diamonds should be replaced by the following condition:

> (v′) Let C be the circle of radius r centered at O, which is tangent to γ_1 at A and B. We denote by $[AP]$ (respectively $[BP]$) the subarc of γ_1 bounded by A (respectively B) and P. We also use another notion $[AP)$ (respectively $[BP)$) which contains the end point A (respectively B) but does not contain the other end point P. Fix $X \in [AP)$ and $Y \in [BP)$. Then there exists a circle C_X (respectively C_Y) of radius δ_μ which is tangent to $[AP)$ at X (respectively $[BP)$ at Y) and $(XP]$ (respectively $(YP]$) lies in Δ_{C_X} (respectively Δ_{C_Y}), where Δ_{C_X} and Δ_{C_Y} are the interior domains of the circles C_X and C_Y, respectively.

From now on, we replace condition (v) with this new modification for the definition of r-diamonds. (All the other conditions (i)–(iv) remain as before. The previous condition (v) is the special case of the new condition (v′) by putting $X = A$ and $Y = B$.) Thus, our r-diamonds satisfy the old definition of r-diamonds in [1], but the converse is not true. By this change, two points A, B can move not only on S_1 but also whole on γ_1 (see [1, (4.9)] for the definition of S_1).

In Claim 1-3 in [1, Section 4], we proved that there is a δ^*_μ-diamond $\diamond_{OAPB}$. This proof can apply for our new definition of r-diamonds by a suitable modification as follows: Since the curvature radius of each point of the spherical curve γ_1 is less than δ_μ (cf. [1, (4.7)]), the condition (v′) holds for the 4-gon $OAPB$ as in [1, Fig. 17] for sufficiently small $r > 0$. By replacing S_1 to γ_1, the proofs of Claim 1-2 are valid even if $A, B \notin S_1$. We now pay attention to the sequence of r_n-diamonds $\{\diamond_{O_nA_nPB_n}\}_{n=1,2,\dots}$ in the proof of Claim 3. For the sake of simplicity, we set $A := A_\infty$ and $B := B_\infty$. From now on, we show that $[AP]$ and $[BP]$ are both simple arcs: In fact, there exist $c_1, c_2 \in S^1 := \boldsymbol{R}/\boldsymbol{Z}$ such that $\gamma_1(c_1) = P$ and $\gamma_1(c_2) = A$. Since $\diamond_{O_nA_nPB_n}$ is a diamond, $[A_nP]$ is a simple arc. So if we suppose that $[AP]$ has a self-intersection, then there exists $m \in (c_1, c_2)$ such that $A = \gamma_1(m)$. Since all of the crossings on γ_1 are transversal, either $\gamma_1(m - \varepsilon)$ or $\gamma_1(m + \varepsilon)$ does not lie in the interior domain $\Delta_{C_{A_n}}$ for a sufficiently small $\varepsilon > 0$ and for sufficiently large n. However, it contradicts the condition (v′) of the diamond $\diamond_{O_nA_nPB_n}$, since $\gamma_1(m)$ lies in $[A_nP)$ for sufficiently large n. Similarly, $[BP]$ is also a simple arc.

We now suppose that the limit 4-gon $O_\infty APB$ is not a ρ-diamond ($\rho := \delta^*_\mu$).

Then, by the condition (v′), there exists $X \in [AP)$ or $Y \in [BP)$ such that either $(XP]$ meets C_X or $(YP]$ meets C_Y. Without loss of generality, we may consider the case $(XP]$ meets C_X. If $X \neq A$, then $X \in (A_nP]$ holds for a sufficiently large n. Then, the property (v′) of the r_n-diamond $\diamond_{OA_nPB_n}$ yields that C_X does not meet $(XP]$, a contradiction.

So we may assume that $X = A$. Now it is sufficient to make a contradiction under the assumption that $(AP]$ meets C_A. Since $(A_nP]$ lies in the domain $\Delta_{C_{A_n}}$, by taking limit

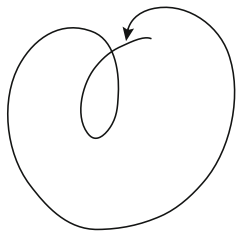

Fig. 4 A figure which should added in [1, Figure 9]

$n \to \infty$, there exists a point Q on $(AP]$ such that Q lies on the circle C_A. Without loss of generality, we may assume that Q is the first such point, namely, (AQ) does not meet C_A. (see [1, Fig. 20, left and right]). We give an orientation of C_A so that it has positive geodesic curvature. Since C_A is of radius δ_μ, the geodesic curvature of γ_1 is greater than that of C_A. Then [1, Lemma 4.5] implies that Q lies in the future part of C_A with respect to A.

We first consider the case $Q = P$. Consider the subarc of C_A defined by

$$\mathfrak{a} := C_A \cap \overline{\Delta_{C_B}}.$$

Then $\mathfrak{a}$ lies in the past part of C_A with respect to A. Since Q lies in the subarc $\mathfrak{a}$, it contradicts the fact that Q lies in the future part of C_A.

We next consider the case $Q \neq P$. We denote by C_A^+ (respectively C_A^-) the future part (respectively the past part) of C_A with respect to A. Since $Q \in C_A^+$, we can consider the subarc $\mathfrak{b}$ of C_A^+ bounded by Q and A. Since $\mathfrak{a} \subset C_A^-$, the two circular arcs $\mathfrak{a}$ and $\mathfrak{b}$ are disjoint. Since the geodesic curvature of $\gamma_1(t)$ at Q is greater than k_0, the subarc $[QP]$ of γ_1 lies on the domain D bounded by $\mathfrak{b}$ and $[AQ)$ (see [1, Fig. 20], right). Since P lies in $\overline{D}$ and $[BP)$ never meets $[AP)$, there is a point R on $[BP) \cap \mathfrak{b}$. On the other hand, since $[BP) \subset \overline{\Delta_{C_B}}$, we have that

$$R \in \mathfrak{a},$$

which contradicts that $\mathfrak{a}$ and $\mathfrak{b}$ are disjoint.

Remark 2.1 In [1, Figure 9], we showed the two possibilities of the behavior of the curve γ_1. However, the case as in Fig. 4 was missing in [1, Figure 9] as the third possibility. Fortunately, this is just the case of Θ_C as in [1, Figure 11], the statement and the proof of [1, Corollary 3.11] themselves do not need any corrections.

3 The proof of Proposition 4.7

For the correction of the proof of Proposition 4.7, we also need the following renewal of the arguments given in [1, Page 131 Line 15–Page 133 Line 21]: We set $\rho := \delta_\mu^*$. Let $\diamond_{OAPB}$ be the ρ-diamond as in Section 2, and C the circle of radius ρ centered at O. By [1, Corollary 3.7], there exists a negative shell S_2 on γ_2. We place S_2 so that it lies inside of the circle C of radius ρ, and the node of S_2 is tangent to the arc $[BP)$ at B (see Fig. 5, left). By Proposition 1.1, S_2 actually lies inside of the circle C. We slide the shell S_2 along the arc $[BP)$. We denote by $S_2(Y)$ this sliding shell whose node lies at Y for each $Y \in [BP)$. When $Y = B$, $S_2(Y)$ coincides with S_2.

Consider the baroon-like closed domain $\bar{\Omega}$ bounded by three arcs $[AP]$, $[BP]$ and AB, where AB is the major arc on C bounded by A, B. By Proposition 1.1, the shell S_2 lies in $\bar{\Omega}$ (cf. Fig. 5, left). Let Z be the first point at which $S_2(Y) \setminus \{Y\}$ meets the boundary of $\bar{\Omega}$.

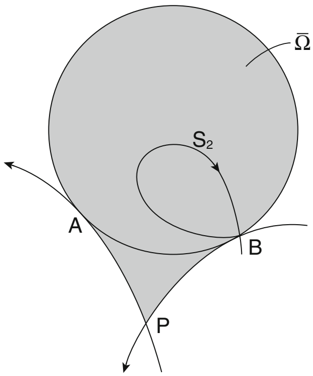

Fig. 5 The sliding operation of S_2

If Z lies on the circle C, then whole of $S_2(Y) \setminus \{Z\}$ lies in Δ_C by Proposition 1.1, which contradicts the fact that $Z \neq Y$ and $Y \in [BP)$. So Z lies in $[AP)$ or $[BP)$. If Z lies in $[AP)$, then (Y, Z) gives an admissible bi-tangent (cf. [1, Definition 2.1.] and Fig. 5, right), which proves Proposition 4.7. So it is sufficient to make a contradiction if Z lies in $[PB)$.

We first consider the case $Z \in (YP]$. By Proposition 1.1, there exists a circle Γ of radius ρ which is tangent to $[BP)$ at Y such that $S_2(Y) \setminus \{Y\}$ lies in Δ_Γ. In particular, $Z \in \Delta_\Gamma$. Since $(YP]$ lies in the interior Δ_{C_Y} of the circle C_Y by the condition (v′) of diamonds, we can conclude that $\Delta_{C_Y} \cap \Delta_\Gamma$ is an empty set. Thus $Z \notin \Delta_{C_Y}$. However, this contradicts the fact that $(YP]$ lies in Δ_{C_Y}.

We next consider the other case $Z \in [BY)$. By Proposition 1.1, there exists a circle Γ' of radius ρ which is tangent to $[BP)$ at Z such that $S_2(Y) \setminus \{Z\}$ lies in the domain $\Delta_{\Gamma'}$. In particular, $Y \in \Delta_{\Gamma'}$. Since $(ZP]$ is contained in Δ_{C_Z} by the condition (v′) of diamonds, $\Delta_{C_Z} \cap \Delta_{\Gamma'}$ is an empty set. Thus $Y \notin \Delta_{C_Z}$, which contradicts the fact that $(ZP]$ lies in Δ_{C_Z}.

All of the other arguments in [1] are correct without any modifications.

Reference

1. Kitagawa, Y., Umehara, M.: Extrinsic diameter of immersed flat tori in S^3. Geom. Dedicata **155**, 105–140 (2011)