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Generalized adjoint forms on algebraic varieties

Luca Rizzi[1] · Francesco Zucconi[1]

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Abstract We prove a full generalization of the Castelnuovo's free pencil trick. We show its analogies with Rizzi and Zucconi (Differential forms and quadrics of the canonical image. arXiv:1409.1826, Theorem 2.1.7); see also Pirola and Zucconi (J Algebraic Geom 12(3):535–572, Theorem 1.5.1). Moreover we find a new formulation of the Griffiths's infinitesimal Torelli Theorem for smooth projective hypersurfaces using meromorphic 1-forms.

Keywords Extension class of a vector bundle · Torsion freeness · Castelnuovo's free pencil trick · Infinitesimal Torelli problem · Projective hypersurface · Meromorphic forms

Mathematics Subject Classification 14C34 · 14D07 · 14J10 · 14J40 · 14J70

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✉ Luca Rizzi
rizzi.luca@spes.uniud.it

Francesco Zucconi
francesco.zucconi@uniud.it

[1] D.I.M.I., The University of Udine, 33100 Udine, Italy

1 Introduction

Let X be an m-dimensional smooth projective variety and $\mathcal{F}$ be a rank n locally free sheaf over it. A way to study $\mathcal{F}$ is to study its extensions $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ which, up to isomorphism, are parametrized by $\text{Ext}^1(\mathcal{F}, \mathcal{L})$. In [2,3,5,6,10–13] and [1] the adjoint forms associated to $\xi \in \text{Ext}^1(\mathcal{O}_X, \mathcal{F})$ are deeply studied and many applications are given. Let us recall the notion of adjoint form in the case $\mathcal{L} = \mathcal{O}_X$.

Given $\xi \in \text{Ext}^1(\mathcal{O}_X, \mathcal{F})$, take an $(n+1)$-dimensional subspace W of the kernel of the cup-product homomorphism $\partial_\xi \colon H^0(X, \mathcal{F}) \to H^1(X, \mathcal{O}_X)$. Denote by $\lambda^i W$ the image of $\bigwedge^i W$ through the natural homomorphism $\lambda^i \colon \bigwedge^i H^0(X, \mathcal{F}) \to H^0(X, \bigwedge^i \mathcal{F})$. If $\mathcal{B} := \langle \eta_1, \ldots, \eta_{n+1} \rangle$ is a basis of W and $s_1, \ldots, s_{n+1} \in H^0(X, \mathcal{E})$ are liftings of $\eta_1, \ldots, \eta_{n+1}$, respectively, then the map $\Lambda^{n+1} \colon \bigwedge^{n+1} H^0(X, \mathcal{E}) \to H^0(X, \bigwedge^{n+1} \mathcal{E})$ gives the top form $\Omega := \Lambda^{n+1}(s_1 \wedge s_2 \wedge \ldots \wedge s_{n+1}) \in H^0(X, \det \mathcal{E})$. The section Ω corresponds to a top form $\omega_{\xi, W, \widehat{\mathcal{B}}} \in H^0(X, \det \mathcal{F})$ via the isomorphism $\det \mathcal{F} \simeq \det \mathcal{E}$, where $\widehat{\mathcal{B}} = \langle s_1, \ldots, s_{n+1} \rangle$; the form $\omega_{\xi, W, \widehat{\mathcal{B}}}$ is called *an adjoint form of W and ξ*. To the basis $\mathcal{B}$ there are also naturally associated $n+1$ elements $\omega_i := \lambda^n(\eta_1 \wedge \ldots \wedge \eta_{i-1} \wedge \widehat{\eta_i} \wedge \eta_{i+1} \wedge \ldots \wedge \eta_{n+1})$, $i = 1, \ldots, n+1$, obtained by the basis $\langle \eta_1 \wedge \ldots \wedge \eta_{i-1} \wedge \widehat{\eta_i} \wedge \eta_{i+1} \wedge \ldots \wedge \eta_{n+1} \rangle_{i=1}^{n+1}$ of $\bigwedge^n W$. Note that if we change the liftings $s_1, \ldots, s_{n+1} \in H^0(X, \mathcal{E})$ with other liftings $\widetilde{s}_1, \ldots, \widetilde{s}_{n+1}$, then $\omega_{\xi, W, \widehat{\mathcal{B}}}$ is a linear combination of $\omega_{\xi, W, \widetilde{\mathcal{B}}}$ and $\omega_1, \ldots, \omega_{n+1}$. The natural problem of this theory is to characterize the condition $\omega_{\xi, W, \widehat{\mathcal{B}}} \in \lambda^n W$ in terms of the fixed divisor D_W of $|\lambda^n W| \subset \mathbb{P}H^0(X, \det \mathcal{F})$ and of the base locus Z_W of the moving part $M_W \in \mathbb{P}H^0(X, \det \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D_W))$, where $|\lambda^n W| = D_W + |M_W|$.

In this paper we consider the general case where $\mathcal{L}$ is an invertible sheaf not necessarily equal to $\mathcal{O}_X$. In this case $\det \mathcal{E} = \mathcal{L} \otimes \det \mathcal{F}$ and liftings $s_1, \ldots, s_{n+1} \in H^0(X, \mathcal{E})$ of $\eta_1, \ldots, \eta_{n+1} \in H^0(X, \mathcal{F})$ determine $\Omega := \Lambda^{n+1}(s_1 \wedge s_2 \wedge \ldots \wedge s_{n+1}) \in H^0(X, \det \mathcal{E})$ which is now called a generalized adjoint form. We define as before $\omega_i := \lambda^n(\eta_1 \wedge \ldots \wedge \eta_{i-1} \wedge \widehat{\eta_i} \wedge \eta_{i+1} \wedge \ldots \wedge \eta_{n+1})$, $i = 1, \ldots, n+1$ and we characterize the case where Ω belongs to the image of $H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E})$ by the natural tensor product map. The game is more complicated than in the above-mentioned papers because the linear system $|\lambda^n W|$ is inside $\mathbb{P}H^0(X, \det \mathcal{F})$ and we have to relate the fixed divisor D_W of $|\lambda^n W|$ and the base locus Z_W of the moving part $M_W \in \mathbb{P}H^0(X, \det \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D_W))$ to forms which are not anymore inside $H^0(X, \det \mathcal{F})$. Nevertheless the result is analogue to the one of [11, Theorem 1.5.1] and [13, Theorem 2.1.7]:

Theorem [A] *Let X be an m-dimensional complex compact smooth variety. Let $\mathcal{F}$ be a rank n locally free sheaf on X and $\mathcal{L}$ an invertible sheaf. Consider an extension $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ corresponding to $\xi \in \text{Ext}^1(\mathcal{F}, \mathcal{L})$. Let $W = \langle \eta_1, \ldots, \eta_{n+1} \rangle$ be an $n+1$-dimensional sublinear system of $\ker(\partial_\xi) \subset H^0(X, \mathcal{F})$. Let $\Omega \in H^0(X, \det \mathcal{E})$ be a generalized adjoint form associated to W as above. It holds that if $\Omega \in \text{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$ then $\xi \in \ker(H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}) \to H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}(D_W)))$.*

Theorem [A], called Adjoint Theorem, can be thought as a general version of the well-known Castelnuovo's free pencil trick; c.f. see Theorem 2.8.

We have also a viceversa of the Adjoint Theorem; see: Theorem 2.9:

Theorem [B] *Under the same hypothesis of Theorem [A], assume also that $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$. It holds that if $\xi \in \ker(H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}) \to H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}(D_W)))$, then $\Omega \in \text{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$.*

In particular in the case $D_W = 0$ Theorem [B] is a full characterization of the condition $\Omega \in \text{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$.

Now by the Adjoint Theorem and by Theorem [B] we can study extension classes of sheaves via adjoint forms. Indeed even if $\mathcal{F}$ has no global sections we can always take the tensor product with a sufficiently ample linear system $\mathcal{M}$ such that $\mathcal{F}\otimes\mathcal{M}$ has enough global sections in order to apply the theory of adjoint forms. By applying the above idea to the case where $n>2$, $X\subset\mathbb{P}^n$ is an hypersurface of degree $d>3$ and $\mathcal{F}:=\Omega^1_X\otimes_{\mathcal{O}_X}\mathcal{O}_X(2)$ we have a reformulation of the infinitesimal Torelli Theorem for X in the setting of generalized adjoint theory. In this paper we will not recall the theory concerning infinitesimal Torelli Theorems, for which a reference is [16], in any case a quick introduction to this topic is also given in [13]. Here we point out only that given a degree d form $F\in\mathbb{C}[\xi_0,\dots,\xi_n]$ the Jacobian ideal of F is the ideal $\mathcal{J}$ generated by the partial derivatives $\frac{\partial F}{\partial \xi_i}$ for $i=0,\dots,n$ and by [9][Theorem 9.8], any infinitesimal deformation $\xi\in H^1(X,\Theta_X)$, where $X=(F=0)$ and Θ_X is the sheaf of tangent vectors on X, is given by a class $[R]$ in the quotient $\mathbb{C}[\xi_0,\dots,\xi_n]/\mathcal{J}$ where R is a homogeneous form of degree d.

Theorem [C] *For a smooth hypersurface X of degree d in $\mathbb{P}^n$ with $n\geq 3$ and $d>3$ the following are equivalent:*

(1) *the differential of the period map is zero on the infinitesimal deformation*

$$[R]\in(\mathbb{C}[\xi_0,\dots,\xi_n]/\mathcal{J})_d\simeq H^1(X,\Theta_X)$$

(2) *R is an element of the Jacobian ideal $\mathcal{J}$*

(3) $\Omega\in\operatorname{Im}(H^0(X,\mathcal{O}_X(2))\otimes\lambda^nW\to H^0(X,\mathcal{O}_X(n+d-1)))$ *for the generic generalized adjoint Ω*

(4) *The generic generalized adjoint Ω lies in $\mathcal{J}$.*

Note that Theorem [C] has a different flavor with respect to the analogue [9, Theorem 9.8] since we essentially use meromorphic 1-forms over X; see Proposition 3.7. Finally we want to mention that in a forthcoming paper [14] we show how to recover also the Green's infinitesimal Torelli Theorem for a sufficiently ample divisor of a smooth variety in terms of generalized adjoint theory.

2 The theory of generalized adjoint forms

2.1 Definition of generalized adjoint form

Let X be a smooth compact complex variety of dimension m and let $\mathcal{F}$ and $\mathcal{L}$ be two locally free sheaves on X of rank n and 1, respectively. Consider the exact sequence of locally free sheaves

$$0\to\mathcal{L}\to\mathcal{E}\to\mathcal{F}\to 0 \tag{2.1}$$

associated to an element $\xi\in\operatorname{Ext}^1(\mathcal{F},\mathcal{L})\cong H^1(X,\mathcal{F}^\vee\otimes\mathcal{L})$. Recall that the invertible sheaf $\det\mathcal{F}:=\bigwedge^n\mathcal{F}$ fits into the exact sequence

$$0\to\bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L}\to\bigwedge^{n}\mathcal{E}\to\det\mathcal{F}\to 0, \tag{2.2}$$

which still corresponds to ξ under the isomorphism $\operatorname{Ext}^1(\mathcal{F},\mathcal{L})\cong\operatorname{Ext}^1(\det\mathcal{F},\bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L})\cong H^1(X,\mathcal{F}^\vee\otimes\mathcal{L})$. Furthermore $\det\mathcal{F}$ satisfies

$$\det\mathcal{F}\otimes\mathcal{L}\cong\det\mathcal{E}. \tag{2.3}$$

Let $\partial_\xi\colon H^0(X,\mathcal{F}) \to H^1(X,\mathcal{L})$ be the connecting homomorphism related to (2.1), and let $W \subset \ker(\partial_\xi)$ be a vector subspace of dimension $n+1$. Choose a basis $\mathcal{B} := \{\eta_1, \dots, \eta_{n+1}\}$ of W. By definition we can take liftings $s_1, \dots, s_{n+1} \in H^0(X,\mathcal{E})$ of the sections $\eta_1, \dots, \eta_{n+1}$. If we consider the natural map

$$\Lambda^n\colon \bigwedge^n H^0(X,\mathcal{E}) \to H^0(X, \bigwedge^n \mathcal{E})$$

we can define the sections

$$\Omega_i := \Lambda^n(s_1 \wedge \dots \wedge \hat{s_i} \wedge \dots \wedge s_{n+1}) \tag{2.4}$$

for $i = 1, \dots, n+1$. Denote by ω_i, for $i = 1, \dots, n+1$, the corresponding sections in $H^0(X, \det\mathcal{F})$. Obviously we have that $\omega_i = \lambda^n(\eta_1 \wedge \dots \wedge \hat{\eta_i} \wedge \dots \wedge \eta_{n+1})$, where λ^n is the natural morphism

$$\lambda^n\colon \bigwedge^n H^0(X,\mathcal{F}) \to H^0(X, \det\mathcal{F}).$$

The vector subspace of $H^0(X, \det\mathcal{F})$ generated by $\omega_1, \dots, \omega_{n+1}$ is denoted by $\lambda^n W$.

Definition 2.1 If $\lambda^n W$ is nontrivial, it induces a sublinear system $|\lambda^n W| \subset \mathbb{P}(H^0(X, \det\mathcal{F}))$ that we will call *adjoint sublinear system*. We call D_W its fixed divisor and Z_W the base locus of its moving part $|M_W| \subset \mathbb{P}(H^0(X, \det\mathcal{F}(-D_W)))$.

Definition 2.2 The section $\Omega \in H^0(X, \det\mathcal{E})$ corresponding to $s_1 \wedge \dots \wedge s_{n+1}$ via

$$\Lambda^{n+1}\colon \bigwedge^{n+1} H^0(X,\mathcal{E}) \to H^0(X, \det\mathcal{E}) \tag{2.5}$$

is called generalized adjoint form.

Remark 2.3 It is easy to see by local computation that this section is in the image of the natural injection $\det\mathcal{E}(-D_W) \otimes \mathcal{I}_{Z_W} \to \det\mathcal{E}$.

We want to study the condition

$$\Omega \in \mathrm{Im}\,(H^0(X,\mathcal{L}) \otimes \langle\Omega_i\rangle \to H^0(X, \det\mathcal{E})) \tag{2.6}$$

or, equivalently,

$$\Omega \in \mathrm{Im}\,(H^0(X,\mathcal{L}) \otimes \lambda^n W \to H^0(X, \det\mathcal{E})). \tag{2.7}$$

The first map is given by the wedge product, the second one by (2.3). Note that if $H^0(X,\mathcal{L}) = 0$ this condition is equivalent to $\Omega = 0$.

Remark 2.4 The choice of the liftings is not relevant for this purpose. Take different liftings $s'_1, \dots, s'_{n+1} \in H^0(X,\mathcal{E})$ of $\eta_1, \dots, \eta_{n+1}$ and call $\Omega'_i \in H^0(X, \bigwedge^n \mathcal{E})$ and $\Omega' \in H^0(X, \det\mathcal{E})$ the corresponding sections constructed as above. Obviously

$$\mathrm{Im}\,(H^0(X,\mathcal{L}) \otimes \langle\Omega_i\rangle \to H^0(X, \det\mathcal{E})) = \mathrm{Im}\,(H^0(X,\mathcal{L}) \otimes \langle\Omega'_i\rangle \to H^0(X, \det\mathcal{E})), \tag{2.8}$$

since they are both equal to $\mathrm{Im}\,(H^0(X,\mathcal{L}) \otimes \lambda^n W \to H^0(X, \det\mathcal{E}))$. It is also easy to see that $\Omega \in \mathrm{Im}\,(H^0(X,\mathcal{L}) \otimes \langle\Omega_i\rangle \to H^0(X, \det\mathcal{E}))$ iff $\Omega' \in \mathrm{Im}\,(H^0(X,\mathcal{L}) \otimes \langle\Omega'_i\rangle \to H^0(X, \det\mathcal{E}))$.

Remark 2.5 Consider another basis $\mathcal{B}' := \{\eta'_1, \ldots, \eta'_{n+1}\}$ of W and let A be the matrix of the basis change. The sections $s'_1, \ldots, s'_{n+1}$ obtained from $s_1, \ldots, s_{n+1}$ through the matrix A are liftings of $\eta'_1, \ldots, \eta'_{n+1}$. The section $\Omega' := \Lambda^{n+1}(s'_1 \wedge \ldots \wedge s'_{n+1})$ satisfies $\Omega' = \det A \cdot \Omega$. Moreover $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E}))$ iff $\Omega' \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \langle \Omega'_i \rangle \to H^0(X, \det \mathcal{E}))$.

Lemma 2.6 *If* $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \langle \Omega_i \rangle \to H^0(X, \det \mathcal{E}))$, *then we can find liftings* $\tilde{s_i} \in H^0(X, \mathcal{E})$, $i = 1, \ldots, n+1$, *such that* $\tilde{\Omega} := \Lambda^{n+1}(\tilde{s_1} \wedge \ldots \wedge \tilde{s}_{n+1}) = 0$.

Proof By hypothesis there exist $\sigma_i \in H^0(X, \mathcal{L})$ such that

$$\Omega = \sum_{i=1}^{n+1} \sigma_i \wedge \Omega_i \tag{2.9}$$

We can define new liftings for the element η_i:

$$\tilde{s_i} := s_i + (-1)^{n-i} \sigma_i.$$

Now, since

$$\tilde{s_1} \wedge \ldots \wedge \tilde{s}_{n+1} = s_1 \wedge \ldots \wedge s_{n+1} - \sum_{i=1}^{n+1} s_1 \wedge \ldots \wedge \hat{s_i} \wedge \ldots \wedge s_{n+1} \wedge \sigma_i, \tag{2.10}$$

we immediately deduce $\tilde{\Omega} = 0$. □

From the natural map

$$\mathcal{F}^\vee \otimes \mathcal{L} \to \mathcal{F}^\vee \otimes \mathcal{L}(D_W)$$

we have a homomorphism

$$H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}) \xrightarrow{\rho} H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}(D_W));$$

we call $\xi_{D_W} = \rho(\xi)$.

2.2 Castelnuovo's free pencil trick

Consider the case where both $\mathcal{L}$ and $\mathcal{F}$ are of rank one, while X has dimension m. In this case $W = \langle \eta_1, \eta_2 \rangle \subset H^0(X, \mathcal{F})$ has dimension two; as usual we choose liftings $s_1, s_2 \in H^0(X, \mathcal{E})$ of η_1, η_2. Note also that $\omega_1 = \eta_2$ and $\omega_2 = \eta_1$, in particular $W = \lambda^1 W$ so D_W is the fixed part of W and Z_W is the base locus of its moving part. Call $\tilde{\eta}_i \in H^0(X, \mathcal{F}(-D_W))$ the sections corresponding to the η_i's via $H^0(X, \mathcal{F}(-D_W)) \to H^0(X, \mathcal{F})$. The following lemma is well known and it is the core of the Castelnuovo base point free pencil trick.

Lemma 2.7 *We have an exact sequence*

$$0 \to \mathcal{F}^\vee(D_W) \xrightarrow{i} \mathcal{O}_X \oplus \mathcal{O}_X \xrightarrow{\nu} \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \to 0 \tag{2.11}$$

where the morphism i *is given by contraction with* $-\tilde{\eta}_1$ *and* $\tilde{\eta}_2$, *while* ν *is given by evaluation with* $\tilde{\eta}_2$ *on the first component and* $\tilde{\eta}_1$ *on the second one.*

It is easy to see by local computation that sequence (2.11) fits into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & \mathcal{E}^\vee & \longrightarrow & \mathcal{L}^\vee & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot D_W} & & \downarrow{\scriptstyle (-s_1,s_2)} & & \downarrow{\scriptstyle \Omega} & & \\ 0 & \longrightarrow & \mathcal{F}^\vee(D_W) & \xrightarrow{i} & \mathcal{O}_X \oplus \mathcal{O}_X & \xrightarrow{\nu} & \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} & \longrightarrow & 0. \end{array} \tag{2.12}$$

The morphism $\mathcal{E}^\vee \to \mathcal{O}_X \oplus \mathcal{O}_X$ is given by contraction with the sections $-s_1$ and s_2, the morphism $\mathcal{L}^\vee \to \mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W}$ by contraction with the adjoint Ω. We can prove now the following

Theorem 2.8 *Let X be an m-dimensional complex compact smooth variety. Let $\mathcal{F}$, $\mathcal{L}$ be invertible sheaves on X. Consider $\xi \in H^1(X, \mathcal{F}^\vee \otimes \mathcal{L})$ associated to the extension* (2.1). *Define $W = \langle \eta_1, \eta_2 \rangle \subset \ker(\partial_\xi) \subset H^0(X, \mathcal{F})$ and Ω as above. We have that $\Omega \in \operatorname{Im}(H^0(X, \mathcal{L}) \otimes W \to H^0(X, \det \mathcal{E}))$ if and only if $\xi_{D_W} = 0$.*

Proof Tensoring (2.12) by $\mathcal{L}$ and passing to cohomology we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{F}^\vee \otimes \mathcal{L}) & \longrightarrow & H^0(\mathcal{E}^\vee \otimes \mathcal{L}) & \longrightarrow & \mathbb{C} & \longrightarrow & H^1(\mathcal{F}^\vee \otimes \mathcal{L}) \\ & & \downarrow & & \downarrow{\scriptstyle (s_1,-s_2)} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \rho} \\ 0 & \longrightarrow & H^0(\mathcal{F}^\vee(D_W) \otimes \mathcal{L}) & \xrightarrow{i} & H^0(\mathcal{L} \oplus \mathcal{L}) & \xrightarrow{\nu} & H^0(\mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \mathcal{L}) & \xrightarrow{\delta} & H^1(\mathcal{F}^\vee(D_W) \otimes \mathcal{L}). \end{array} \tag{2.13}$$

Obviously $\beta(1) = \Omega$ and, by commutativity, $\delta(\beta(1)) = \xi_{D_W}$. We have then $\xi_{D_W} = 0$ if and only if $\Omega \in \operatorname{Im}(H^0(\mathcal{L} \oplus \mathcal{L}) \overset{\nu}{\to} H^0(\mathcal{F}(-D_W) \otimes \mathcal{I}_{Z_W} \otimes \mathcal{L}))$. Since ν is given by the sections $\tilde{\eta}_2$ and $\tilde{\eta}_1$, this condition is equivalent to $\Omega \in \operatorname{Im}(H^0(X, \mathcal{L}) \otimes W \to H^0(X, \det \mathcal{E}))$, since $\det \mathcal{E} = \mathcal{F} \otimes \mathcal{L}$. □

2.3 The Adjoint Theorem

We go back now to the general case with $\mathcal{F}$ locally free of rank n. By obvious identifications the natural map

$$\operatorname{Ext}^1(\det \mathcal{F}, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}) \to \operatorname{Ext}^1(\det \mathcal{F}(-D_W), \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L})$$

gives an extension $\mathcal{E}^{(n)}$ and a commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L} & \longrightarrow & \mathcal{E}^{(n)} & \xrightarrow{\alpha} & \det \mathcal{F}(-D_W) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow & & \\ 0 & \longrightarrow & \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L} & \longrightarrow & \bigwedge^n \mathcal{E} & \longrightarrow & \det \mathcal{F} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \det \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_W} & = & \det \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_W} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array} \tag{2.14}$$

2.3.1 The proof of the Adjoint Theorem

By the hypothesis $\Omega \in \mathrm{Im}\,(H^0(X,\mathcal{L})\otimes\lambda^n W \to H^0(X,\det\mathcal{E}))$ and by lemma (2.6), we can choose liftings $s_i \in H^0(X,\mathcal{E})$ of η_i with $\Omega = 0$.

Since D_W is the fixed divisor of the linear system $|\lambda^n W|$ and the sections ω_i generate this linear system, then the ω_i are in the image of

$$\det\mathcal{F}(-D_W) \to \det\mathcal{F},$$

so we can find sections $\tilde{\omega}_i \in H^0(X,\det\mathcal{F}(-D_W))$ such that

$$\tilde{\omega}_i \cdot d = \omega_i, \tag{2.15}$$

where d is a global section of $\mathcal{O}_X(D_W)$ with $(d) = D_W$. Hence, using the commutativity of (2.14), we can find liftings $\tilde{\Omega}_i \in H^0(X,\mathcal{E}^{(n)})$ of the sections Ω_i. The evaluation map

$$\bigoplus_{i=1}^{n+1}\mathcal{O}_X \stackrel{\tilde{\mu}}{\to} \mathcal{E}^{(n)}$$

given by the global sections $\tilde{\Omega}_i$, composed with the map α of (2.14), induces a map μ which fits into the following diagram

$$\begin{array}{ccccccccc} & & & & \bigoplus_{i=1}^{n+1}\mathcal{O}_X & = & \bigoplus_{i=1}^{n+1}\mathcal{O}_X & & \\ & & & & \downarrow{\scriptstyle\tilde{\mu}} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \longrightarrow & \bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L} & \longrightarrow & \mathcal{E}^{(n)} & \xrightarrow{\alpha} & \det\mathcal{F}(-D_W) & \longrightarrow & 0. \end{array}$$

We point out that the morphism μ is given by multiplication by $\tilde{\omega}_i$ on the i-th component. The sheaf $\mathrm{Im}\,\tilde{\mu}$ is torsion free since it is a subsheaf of the locally free sheaf $\mathcal{E}^{(n)}$. Moreover, since $\Omega = 0$, a local computation shows that $\mathrm{Im}\,\tilde{\mu}$ has rank one outside Z_W. On the other hand the sheaf $\mathrm{Im}\,\mu$ is by definition

$$\mathrm{Im}\,\mu = \det\mathcal{F}(-D_W)\otimes\mathcal{I}_{Z_W}.$$

The morphism

$$\alpha\colon \mathcal{E}^{(n)} \to \det\mathcal{F}(-D_W)$$

induces a surjective morphism, that we continue to call α,

$$\mathrm{Im}\,\tilde{\mu} \stackrel{\alpha}{\to} \mathrm{Im}\,\mu$$

between two sheaves that are locally free of rank one outside Z_W. This morphism is also injective, because its kernel is a torsion subsheaf of the torsion free sheaf $\mathrm{Im}\,\tilde{\mu}$, hence it is trivial.

We have proved that

$$\mathrm{Im}\,\tilde{\mu} \cong \det\mathcal{F}(-D_W)\otimes\mathcal{I}_{Z_W},$$

so

$$\mathcal{E}^{(n)} \supset (\mathrm{Im}\,\tilde{\mu})^{\vee\vee} \cong \det\mathcal{F}(-D_W).$$

This isomorphism gives the splitting

$$0 \longrightarrow \bigwedge^{n-1}\mathcal{F}\otimes\mathcal{L} \longrightarrow \mathcal{E}^{(n)} \longrightarrow \det\mathcal{F}(-D_W) \longrightarrow 0.$$

Since ξ_{D_W} is the element of $H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}(D_W))$ associated to this extension, we conclude that $\xi_{D_W} = 0$.

We have proved the Adjoint Theorem.

2.3.2 An inverse of the Adjoint Theorem

We prove now an inverse of the Adjoint Theorem.

Theorem 2.9 *Let X be an m-dimensional complex compact smooth variety. Let $\mathcal{F}$ be a rank n locally free sheaf on X and $\mathcal{L}$ an invertible sheaf. Consider an extension $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ corresponding to $\xi \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{L})$. Let $W = \langle \eta_1, \ldots, \eta_{n+1} \rangle$ be a $n+1$-dimensional sublinear system of $\ker(\partial_\xi) \subset H^0(X, \mathcal{F})$. Let $\Omega \in H^0(X, \det \mathcal{E})$ be an adjoint form associated to W as above. Assume that $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$. If $\xi \in \ker(H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}) \to H^1(X, \mathcal{F}^\vee \otimes \mathcal{L}(D_W)))$, then $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$.*

Proof If $\mathcal{F}$ is a rank one sheaf, then (2.8) gives the thesis without the extra assumption $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$. We assume then rank $\mathcal{F} \geq 2$.

By (2.3), we can write $(\Omega) = D_W + F$ with F effective. In the first step of the proof we want to find a global section

$$\Omega' \in H^0\left(X, \bigwedge^n \mathcal{E} \otimes \mathcal{L}(-F)\right)$$

which restricts, through the natural map

$$\bigwedge^n \mathcal{E} \otimes \mathcal{L}(-F) \to \det \mathcal{E}(-F),$$

to the section $d \in H^0(\det \mathcal{E}(-F))$, where $(d) = D_W$.

Consider the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2}(-F) & \to & \bigwedge^n \mathcal{E} \otimes \mathcal{L}(-F) & \xrightarrow{G_2} & \det \mathcal{E}(-F) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle G_1} & & \downarrow & & \\
0 & \to & \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2} & \xrightarrow{\tau} & \bigwedge^n \mathcal{E} \otimes \mathcal{L} & \to & \det \mathcal{E} & \to & 0 \\
 & & \downarrow{\scriptstyle H_1} & & \downarrow & & \downarrow & & \\
0 & \to & \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2}|_F & \xrightarrow{H_2} & \bigwedge^n \mathcal{E} \otimes \mathcal{L}|_F & \xrightarrow{H_3} & \det \mathcal{E}|_F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

By the hypothesis $\xi_{D_W} = 0$ it follows easily that there exists a lifting $\tilde{\Omega} \in H^0(X, \bigwedge^n \mathcal{E} \otimes \mathcal{L})$ of Ω. Indeed, tensor (2.14) by $\mathcal{L}$ and take a global lifting $f \in H^0(X, \det \mathcal{E}(-D_W))$ of Ω. Since $\xi_{D_W} = 0$, f can be lifted to a section $e \in H^0(X, \mathcal{E}^{(n)} \otimes \mathcal{L})$. Define $\tilde{\Omega} := \psi(e)$. By commutativity, $H_3(\tilde{\Omega}|_F) = 0$ hence we call $\bar{\mu} \in H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2}|_F)$ the lifting of $\tilde{\Omega}|_F$. A local computation shows that the connecting homomorphism

$$\delta\colon H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2}|_F) \to H^1(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2}(-F))$$

maps $\bar{\mu}$ to ξ_{D_W}, which is zero by hypothesis. Then there exists a global section

$$\mu \in H^0(X, \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{L}^{\otimes 2})$$

which is a lifting of $\bar{\mu}$. The section

$$\hat{\Omega} := \Omega - \tau(\mu) \in H^0(X, \bigwedge^{n} \mathcal{E} \otimes \mathcal{L})$$

is a new lifting of Ω that, by construction, vanishes when restricted to F. We call

$$\Omega' \in H^0(X, \bigwedge^{n} \mathcal{E} \otimes \mathcal{L}(-F))$$

the global section which lifts $\hat{\Omega}$. It is easy to see that $G_2(\Omega') = d$ so Ω' is the section we wanted.

In the second part of the proof we prove that $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$. The global sections

$$\omega_i := \lambda^n(\eta_1 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \eta_{n+1}) \in H^0(X, \det \mathcal{F})$$

generate $\lambda^n W$ and by definition they vanish on D_W, that is there exist global sections $\tilde{\omega}_i \in H^0(X, \det \mathcal{F}(-D_W))$ such that

$$\omega_i = \tilde{\omega}_i \cdot d.$$

We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}(-F) & \xrightarrow{\alpha} & W \otimes \mathcal{O}_X & \xrightarrow{\gamma} & \bar{\mathcal{F}} & \longrightarrow & 0 \\ & & \downarrow{\cdot F} & & \downarrow{\beta} & & \downarrow{\iota} & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0. \end{array} \tag{2.16}$$

The map β is locally defined by

$$(f_1, \ldots, f_{n+1}) \mapsto (-1)^n f_1 \cdot s_1 + \cdots + f_{n+1} \cdot s_{n+1}.$$

The map α is defined in the following way: if $f \in \mathcal{L}(-F)(U)$ is a local section, then, locally on U, α is given by

$$f \mapsto (\tilde{\omega}_1(f), \ldots, \tilde{\omega}_{n+1}(f)),$$

where we observe that the sections $\tilde{\omega}_i$ are global sections of the dual sheaf of $\mathcal{L}(-F)$. The sheaf $\bar{\mathcal{F}}$ is by definition the cokernel of the first row. Now, tensoring by $\mathcal{L}^\vee$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-F) & \xrightarrow{\alpha} & W \otimes \mathcal{L}^\vee & \xrightarrow{\gamma} & \bar{\mathcal{F}} \otimes \mathcal{L}^\vee & \longrightarrow & 0 \\ & & \downarrow{\cdot F} & & \downarrow{\beta} & & \downarrow{\iota} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{E} \otimes \mathcal{L}^\vee & \longrightarrow & \mathcal{F} \otimes \mathcal{L}^\vee & \longrightarrow & 0. \end{array} \tag{2.17}$$

Dualizing and tensoring again by $\mathcal{O}_X(D_W)$, we obtain the commutative square

$$\begin{array}{ccc}\bigwedge^n W \otimes \mathcal{L}(D_W) & \xrightarrow{\alpha^\vee} & \det \mathcal{E} \\ \uparrow{\scriptstyle \beta^\vee} & & \uparrow{\scriptstyle \cdot F} \\ \mathcal{E}^\vee \otimes \mathcal{L}(D_W) & \longrightarrow & \mathcal{O}_X(D_W),\end{array}$$

where we have used the isomorphism of vector spaces $W^\vee \cong \bigwedge^n W$, given by

$$\eta^i \mapsto \eta_1 \wedge \ldots \wedge \hat{\eta}_i \wedge \ldots \wedge \eta_{n+1} =: e_i$$

where $\eta^1, \ldots, \eta^{n+1}$ is the basis of $W^\vee$ dual to the basis $\eta_1, \ldots, \eta_{n+1}$ of W. By definition of α we have that $\alpha^\vee$ is the evaluation map given by the global sections $\tilde{\omega}_i$. Note that $\mathcal{E}^\vee \otimes \mathcal{L}(D_W) \cong \bigwedge^n \mathcal{E} \otimes \mathcal{L}(-F)$. Taking global sections we have

$$\begin{array}{ccc}\bigwedge^n W \otimes H^0(X, \mathcal{L}(D_W)) & \xrightarrow{\overline{\alpha^\vee}} & H^0(X, \det \mathcal{E}) \\ \uparrow{\scriptstyle \overline{\beta^\vee}} & & \uparrow{\scriptstyle \cdot F} \\ H^0(X, \mathcal{E}^\vee \otimes \mathcal{L}(D_W)) & \longrightarrow & H^0(X, \mathcal{O}_X(D_W)).\end{array}$$

The section $\Omega' \in H^0(X, \mathcal{E}^\vee \otimes \mathcal{L}(D_W))$ produces in $H^0(X, \det \mathcal{E})$ the adjoint Ω, so by commutativity

$$\Omega = \overline{\alpha^\vee}(\overline{\beta^\vee}(\Omega')).$$

We have

$$\overline{\beta^\vee}(\Omega') = \sum_{i=1}^{n+1} c_i \cdot e_i \otimes \sigma_i,$$

where $c_i \in \mathbb{C}$ and $\sigma_i \in H^0(X, \mathcal{L}(D_W))$. By our hypothesis $H^0(X, \mathcal{L}) \cong H^0(X, \mathcal{L}(D_W))$, there exists sections $\tilde{\sigma_i} \in H^0(X, \mathcal{L})$ with $\sigma_i = \tilde{\sigma_i} \cdot d$. So

$$\Omega = \overline{\alpha^\vee}(\overline{\beta^\vee}(\Omega')) = \overline{\alpha^\vee}(\sum_{i=1}^{n+1} c_i \cdot e_i \otimes \sigma_i) = \sum_{i=1}^{n+1} c_i \cdot \tilde{\omega}_i \cdot \sigma_i = \sum_{i=1}^{n+1} c_i \cdot \tilde{\omega}_i \cdot d \cdot \tilde{\sigma}_i = \sum_{i=1}^{n+1} c_i \cdot \omega_i \cdot \tilde{\sigma}_i.$$

This is exactly our thesis. □

By the Adjoint Theorem and (2.9) we deduce the following

Corollary 2.10 *If* $D_W = 0$, *then* $\xi = 0$ *iff* $\Omega \in \mathrm{Im}\,(H^0(X, \mathcal{L}) \otimes \lambda^n W \to H^0(X, \det \mathcal{E}))$.

3 Infinitesimal Torelli Theorem for projective hypersurfaces

In this section we want to study adjoint images in the case of smooth hypersurfaces of the projective space $\mathbb{P}^n$.

3.1 Meromorphic 1-forms on a smooth projective hypersurface

Let $V \subset \mathbb{P}^n$ be a smooth hypersurface defined by a homogeneous polynomial $F \in \mathbb{C}[\xi_0, \dots, \xi_n]$ of degree $\deg F = d$. An infinitesimal deformation $\xi \in \mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ of V gives an exact sequence for the sheaf of differential forms Ω^1_V:

$$0 \to \mathcal{O}_V \to \Omega^1_{\mathcal{V}|V} \to \Omega^1_V \to 0. \tag{3.1}$$

We assume that $n \geq 3$, hence $H^0(V, \Omega^1_V) = 0$ and we can not construct the adjoint of this sequence directly, so we twist (3.1) by a suitable integer a such that $\Omega^1_V(a)$ has at least $n = \mathrm{rank}\,(\Omega^1_V) + 1$ global sections. A standard computation shows that $a = 2$ is enough for this purpose, so from now on we will consider the sequence

$$0 \to \mathcal{O}_V(2) \to \Omega^1_{\mathcal{V}|V}(2) \to \Omega^1_V(2) \to 0 \tag{3.2}$$

which again corresponds to $\xi \in \mathrm{Ext}^1(\Omega^1_V(2), \mathcal{O}_V(2)) \cong \mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \cong H^1(V, \Theta_V)$, where Θ_V denotes the sheaf of vector fields on V. Denote by $\mathcal{J}$ the Jacobian ideal of F, that is the ideal of $\mathbb{C}[\xi_0, \dots, \xi_n]$ generated by the partial derivatives $\frac{\partial F}{\partial \xi_i}$ for $i = 0, \dots, n$. Following [9][Theorem 9.8], the deformation ξ is given by a class $[R]$ of degree d in the quotient $\mathbb{C}[\xi_0, \dots, \xi_n]/\mathcal{J}$. If we choose a representative R of degree d for this class, then $F + tR = 0$, for small t, is the equation of the hypersurface that is the associated deformation of V.

Together with (3.2), we have the conormal exact sequence

$$0 \to \mathcal{O}_V(-d) \to \Omega^1_{\mathbb{P}^n|V} \to \Omega^1_V \to 0. \tag{3.3}$$

If we put these sequences together we obtain the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \Omega^1_{\mathcal{V}|V}(2) & \longrightarrow & \Omega^1_V(2) & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & \Omega^1_{\mathbb{P}^n|V}(2) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & \mathcal{O}_V(2-d) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

which can be completed as follows

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \Omega^1_{\mathcal{V}|V}(2) & \longrightarrow & \Omega^1_V(2) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \mathcal{G} & \longrightarrow & \Omega^1_{\mathbb{P}^n|V}(2) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_V(2-d) & = & \mathcal{O}_V(2-d) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array}
\tag{3.4}
$$

By [9] the deformation ξ of (3.2) comes from $R \in H^0(\mathbb{P}^n, \mathcal{O}^n_{\mathbb{P}}(d))$, then it gives the zero element in $H^0(V, \Theta_{\mathbb{P}^n|V})$, hence we have that the sheaf $\mathcal{G}$ in (3.4) is a direct sum $\mathcal{G} = \mathcal{O}_V(2) \oplus \Omega^1_{\mathbb{P}^n|V}(2)$ and we have a natural morphism $\phi\colon \Omega^1_{\mathbb{P}^n|V}(2) \to \Omega^1_{\mathcal{V}|V}(2)$ which fits in the diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \Omega^1_{\mathcal{V}|V}(2) & \longrightarrow & \Omega^1_V(2) & \longrightarrow & 0 \\
 & & \| & & \uparrow & \overset{\phi}{\nwarrow} & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_V(2) & \longrightarrow & \mathcal{G} & \longrightarrow & \Omega^1_{\mathbb{P}^n|V}(2) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_V(2-d) & = & \mathcal{O}_V(2-d) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array}
\tag{3.5}
$$

The morphism ϕ gives in a natural way a morphism

$$
\begin{aligned}
\phi^n\colon H^0(V, \det(\Omega^1_{\mathbb{P}^n|V}(2))) &\cong H^0(V, \mathcal{O}_V(n-1)) \to H^0(V, \det(\Omega^1_{\mathcal{V}|V}(2))) \\
&\cong H^0(V, \mathcal{O}_V(n+d-1)).
\end{aligned}
$$

We can write explicitly the isomorphism between $H^0(V, \det(\Omega^1_{\mathbb{P}^n|V}(2))) = H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n))$ and $H^0(V, \mathcal{O}_V(n-1))$. Note that $H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(2n)) \to H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n))$ is surjective, so we will focus on the rational n-forms on $\mathbb{P}^n$. By [9][Corollary 2.11] this forms may be written as $\omega = \frac{P\Psi}{Q}$ where $\Psi = \sum_{i=0}^n (-1)^i \xi_i (d\xi_0 \wedge \ldots \wedge \widehat{d\xi_i} \wedge \ldots \wedge d\xi_n)$ gives a generator of $H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(n+1))$ and $\deg Q = \deg P + (n+1)$. In our case Q is a polynomial of degree $2n$, hence P has degree $n-1$. This identification depends on the (noncanonical) choice of the polynomial Q and gives an isomorphism $H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n)) \to H^0(V, \mathcal{O}_V(n-1))$ defined by $\omega|_V \mapsto P$.

Proposition 3.1 *ϕ^n is given via the multiplication by the polynomial R (modulo F).*

Proof Locally we can see $\mathcal{V}$ in the product $\Delta \times \mathbb{P}^n$ of the projective space with a disk; here $\mathcal{V}$ is defined by the equation $F + tR = 0$. Hence $d(F + tR) = 0$ in $\Omega^1_{\mathcal{V}}$, that is $dF + dt \cdot R + dR \cdot t = 0$.

Call $F_i := \frac{\partial F}{\partial \xi_i}$. Since V is smooth, there exist i such that $U_i = (F_i \neq 0)$ is a nontrivial open subset; let for example U_1 be nontrivial. Take local coordinates $z_i = \frac{\xi_i}{\xi_0}$ in the open set $(\xi_0 \neq 0) \cap U_1$. Then we have

$$dz_1 = -\frac{Rdt}{F_1} - \frac{tdR}{F_1} - \sum_{i>1} \frac{F_i}{F_1} dz_i \tag{3.6}$$

which gives in V (that is for $t = 0$)

$$dz_1 = -\frac{Rdt}{F_1} - \sum_{i>1} \frac{F_i}{F_1} dz_i \tag{3.7}$$

The image $\phi^n(\omega|_V)$ is then obtained by the substitution of (3.7) in $\frac{P(z)}{Q(z)} dz_1 \wedge \ldots \wedge dz_n$, which is the local form of $\frac{P(\xi)\Psi}{Q(\xi)}$. Hence

$$\frac{P(z)}{Q(z)} dz_1 \wedge \ldots \wedge dz_n = -\frac{P(z)R(z)}{Q(z)F_1(z)} dt \wedge dz_2 \wedge \ldots \wedge dz_n. \tag{3.8}$$

If we homogenize we obtain on U_1

$$\frac{P\Psi}{Q} = -\frac{PR}{QF_1} \sum_{i \neq 1} (-1)^{i-1} \mathrm{sgn}(i-1) \xi_i dt \wedge d\xi_0 \wedge \widehat{d\xi_1} \ldots \wedge \widehat{d\xi_i} \wedge \ldots \wedge d\xi_n$$

Hence

$$\phi^n(\omega|_V) = -\frac{PR}{QF_1} \sum_{i \neq 1} (-1)^{i-1} \mathrm{sgn}(i-1) \xi_i dt \wedge d\xi_0 \wedge \widehat{d\xi_1} \ldots \wedge \widehat{d\xi_i} \wedge \ldots \wedge d\xi_n \tag{3.9}$$

and it is clear that ϕ^n is given by multiplication with R. □

3.2 A canonical choice of adjoints on a hypersurface of degree $d > 2$

We want now to construct adjoint forms associated to the sequence (3.2).

Assume that $n \geq 3$, so that $H^1(V, \mathcal{O}_V(2)) = H^1(V, \mathcal{O}_V(2-d)) = 0$, and we can lift all the global sections of $H^0(V, \Omega^1_V(2))$ both in the horizontal and in the vertical sequence of (3.5).

We take $\eta_1, \ldots, \eta_n \in H^0(V, \Omega^1_V(2))$ global forms and we want to find liftings $s_1, \ldots, s_n \in H^0(V, \Omega^1_{\mathcal{V}|V})$. This can be done since $H^1(V, \mathcal{O}_V(2))$ is zero. A generalized adjoint is then the global section of the sheaf $\det(\Omega^1_{\mathcal{V}|V}(2)) = \mathcal{O}_V(n+d-1)$ given by $\Omega := \Lambda^n(s_1 \wedge \ldots \wedge s_n) \in H^0(V, \det(\Omega^1_{\mathcal{V}|V}(2)))$.

We point out another interesting way to compute this generalized adjoint form using Proposition (3.1).

Consider the sequence (3.3), that is the vertical sequence in (3.5). Since $H^1(V, \mathcal{O}_V(2-d)) = 0$, we can find liftings $\tilde{s_1}, \ldots, \tilde{s_n} \in H^0(V, \Omega^1_{\mathbb{P}^n|V}(2))$ of the sections $\eta_1, \ldots, \eta_n$.

Furthermore they are unique if $d > 2$. We can thus consider the adjoint form associated to (3.3) given by $\widetilde{\Omega} := \Lambda^n(\tilde{s_1} \wedge \ldots \wedge \tilde{s_n})$. This adjoint is independent from the deformation ξ; it depends only on V and its embedding in $\mathbb{P}^n$. If $d > 2$, then $\widetilde{\Omega}$ is unique.

To describe $\widetilde{\Omega}$ explicitly we first consider the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n}(2-d) \to \Omega^1_{\mathbb{P}^n}(2) \to \Omega^1_{\mathbb{P}^n|V}(2) \to 0. \tag{3.10}$$

If $d > 2$, the vanishing of $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2-d))$ and $H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2-d))$ (c.f. Bott Formulas) gives the isomorphism $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)) = H^0(V, \Omega^1_{\mathbb{P}^n|V}(2))$. Hence, the forms $\tilde{s_i}$ are the restriction on V of global rational 1-forms. By [9][Theorem 2.9] we can write

$$\tilde{s_i} = \frac{1}{Q}\sum_{j=0}^{n} L^i_j d\xi_j \tag{3.11}$$

where $\deg Q = 2$ and L^i_j is a homogeneous polynomial of degree 1 which does not contain ξ_j in its expression. Hence

$$\widetilde{\Omega} = \Lambda^n(\tilde{s_1} \wedge \ldots \wedge \tilde{s_n}) = \frac{1}{Q^n}\sum_{i=0}^{n} M_i d\xi_0 \wedge \ldots \wedge \widehat{d\xi_i} \wedge \ldots \wedge d\xi_n \tag{3.12}$$

where M_i is the determinant of the matrix obtained by

$$\begin{pmatrix} L^1_0 & \ldots & L^n_0 \\ \vdots & & \vdots \\ L^1_n & \ldots & L^n_n \end{pmatrix} \tag{3.13}$$

removing the i-th row. Since $\widetilde{\Omega}$ is a rational n-form on $\mathbb{P}^n$, following [9][Corollary 2.11] it can be written as $\frac{P\Psi}{Q^n}$, and we deduce that

$$\frac{M_i}{(-1)^i \xi_i} = P \tag{3.14}$$

for all $i = 0, \ldots, n$. P is a polynomial of degree $n-1$ and it corresponds to $\widetilde{\Omega}$ via the isomorphism $H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n)) \cong H^0(V, \mathcal{O}_V(n-1))$. Hence by (3.1) we have that the form $\Omega \in H^0(V, \mathcal{O}_V(n+d-1))$ given by PR is a canonical choice of adjoint form for $W = \langle \eta_1, \ldots, \eta_n \rangle$ and ξ.

Remark 3.2 Alternatively this can be seen using the Euler sequence on V:

$$0 \to \mathcal{O}_V \to \bigoplus^{n+1} \mathcal{O}_V(1) \to \Theta_{\mathbb{P}^n|V} \to 0. \tag{3.15}$$

This sequence, dualized and conveniently tensorized gives

$$0 \to \Omega^1_{\mathbb{P}^n|V}(2) \to \bigoplus_{i=1}^{n+1} \mathcal{O}_V(1) \to \mathcal{O}_V(2) \to 0. \tag{3.16}$$

The sections $\tilde{s_i}$ are associated via the first morphism to an $n+1$-uple of linear polynomials $(L^0_i, \ldots, L^n_i)$. Then, taking the wedge product of (3.16) we obtain an exact sequence

$$0 \to \Omega^n_{\mathbb{P}^n|V}(2n) \cong \mathcal{O}_V(n-1) \to \bigwedge^n \mathcal{O}_V(1) = \bigoplus^{n+1} \mathcal{O}_V(n) \to \Omega^{n-1}_{\mathbb{P}^n|V}(2n) \to 0 \tag{3.17}$$

where the morphism $\mathcal{O}_V(n-1) \to \bigoplus^{n+1} \mathcal{O}_V(n)$ is given by

$$G \mapsto (G\xi_0, \ldots, (-1)^n G\xi_n). \tag{3.18}$$

Since $\widetilde{\Omega} = \Lambda^n(\tilde{s_1} \wedge \ldots \wedge \tilde{s_n}) \in H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n))$ is sent exactly to $(L_0^0, \ldots, L_0^n) \wedge \ldots \wedge (L_n^0, \ldots, L_n^n) = (M_0, \ldots, M_n)$ (using the same notation as above), then we conclude that $\widetilde{\Omega}$ corresponds in $H^0(V, \mathcal{O}_V(n-1))$ to a polynomial P which satisfies

$$\frac{M_i}{(-1)^i \xi_i} = P. \tag{3.19}$$

3.3 The adjoint sublinear systems obtained by meromorphic 1-forms

To study the conditions given in (2.6) and (2.7), we need to describe the sections

$$\widetilde{\Omega_i} := \Lambda^{n-1}(\tilde{s_1} \wedge \ldots \wedge \hat{\tilde{s_i}} \wedge \ldots \wedge \tilde{s_n}) \in H^0(V, \Omega^{n-1}_{\mathbb{P}^n|V}(2n-2))$$

(c.f. (2.4)) and their images in $H^0(V, \Omega^{n-1}_V(2(n-1))) = H^0(V, \mathcal{O}_V(n+d-3))$ that we have denoted by ω_i.

A computation similar to the above shows that

$$\widetilde{\Omega_i} = \Lambda^{n-1}(\tilde{s_1} \wedge \ldots \wedge \hat{\tilde{s_i}} \wedge \ldots \wedge \tilde{s_n}) = \frac{1}{Q^{n-1}} \sum_{j<k} M^i_{jk} d\xi_0 \wedge \ldots \wedge d\hat{\xi}_j \wedge \ldots \wedge d\hat{\xi}_k \wedge \ldots \wedge d\xi_n \tag{3.20}$$

where M^i_{jk} is the determinant of the matrix obtained by (3.13) removing the i-th column and the j-th and k-th rows. On the other hand, rearranging the expression of [9][Theorem 2.9] we can write

$$\widetilde{\Omega_i} = \frac{1}{Q^{n-1}} \sum_j A^i_j \Big(\sum_{k \neq j} (-1)^{k+j} \operatorname{sgn}(k-j) \xi_k d\xi_0 \wedge \ldots \wedge d\hat{\xi}_j \wedge \ldots \wedge d\hat{\xi}_k \wedge \ldots \wedge d\xi_n\Big) \tag{3.21}$$

with $\deg A^i_j = n-2$.

Comparing (3.20) and (3.21) gives

$$M^i_{jk} = (-1)^{j+k}(A^i_j \xi_k - \xi_j A^i_k). \tag{3.22}$$

As before this can be computed also via the Euler sequence.

We call $\Xi_j := \sum_{k \neq j} (-1)^{k+j} \operatorname{sgn}(k-j) \xi_k d\xi_0 \wedge \ldots \wedge d\hat{\xi}_j \wedge \ldots \wedge d\hat{\xi}_k \wedge \ldots \wedge d\xi_n$. Note that the sections Ξ_j, for $j = 0, \ldots, n$ give a basis of $H^0(V, \Omega^{n-1}_{\mathbb{P}^n|V}(n))$.

Proposition 3.3 *$\omega_i = \sum_j A^i_j \cdot F_j$ in $H^0(V, \mathcal{O}_V(n+d-3))$*

Proof It is enough to show that the image of Ξ_j through the morphism $\Omega^{n-1}_{\mathbb{P}^n|V}(n) \to \mathcal{O}_V(d-1)$ is F_j. Consider the exact sequence of the tangent sheaf of V:

$$0 \to \Theta_V \to \Theta_{\mathbb{P}^n|V} \to \mathcal{O}_V(d) \to 0. \tag{3.23}$$

The beginning of the Koszul complex is

$$\bigwedge^n \Theta_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-d) \to \bigwedge^{n-1} \Theta_{\mathbb{P}^n|V} \tag{3.24}$$

which, tensored by $\mathcal{O}_V(-n)$, gives

$$\bigwedge^n \Theta_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-n-d) \to \bigwedge^{n-1} \Theta_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-n). \tag{3.25}$$

This is exactly the dual of $\Omega^{n-1}_{\mathbb{P}^n|V}(n) \to \mathcal{O}_V(d-1)$. Hence we only need to show that the morphism (3.25) composed with the contraction by Ξ_i

$$\bigwedge^{n-1} \Theta_{\mathbb{P}^n|V} \otimes \mathcal{O}_V(-n) \stackrel{\Xi_i}{\to} \mathcal{O}_V \tag{3.26}$$

is the multiplication by F_i. This is easy to see by a standard local computation. □

Remark 3.4 We immediately have that the polynomials associated to the sections ω_i are in the Jacobian ideal of V.

Condition (2.7), that is

$$\Omega \in \mathrm{Im}\,(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1))), \tag{3.27}$$

can be written, modulo F, as

$$RP = \sum \omega_i \cdot S_i = \sum_{i,j} A^i_j \cdot F_j \cdot S_i, \tag{3.28}$$

where $\deg S_i = 2$. In particular this implies that RP is in the Jacobian ideal of V.

Proposition 3.5 *The base locus D_W of the linear system $|\lambda^n W|$ is zero for the generic W.*

Proof By [11][Proposition 3.1.6] it is enough to prove that $H^0(V, \Omega^1_V(2))$ generically generates the sheaf $\Omega^1_V(2)$ and that $D_{H^0(V,\Omega^1_V(2))} = 0$. We have an explicit basis for $H^0(V, \Omega^1_V(2))$ given by

$$\frac{\xi_i d\xi_j - \xi_j d\xi_i}{Q} \tag{3.29}$$

where $i < j$ and $\deg Q = 2$. The vector space $\lambda^n H^0(V, \Omega^1_V(2)) \subset H^0(V, \mathcal{O}_V(n+d-3))$ is obviously nonzero, hence $H^0(V, \Omega^1_V(2))$ generically generates the sheaf $\Omega^1_V(2)$.

It remains to prove that $D_{H^0(V,\Omega^1_V(2))} = 0$. An easy computation (for example by induction) shows that $\lambda^n H^0(V, \Omega^1_V(2))$ contains all the polynomials of the form

$$\xi_{i_1}\xi_{i_2}\dots\xi_{i_{n-2}}\frac{\partial F}{\partial \xi_j} \tag{3.30}$$

where $\{i_1, \dots, i_{n-2}\} \subset \{1, \dots, n+1\}$ and $j \notin \{i_1, \dots, i_{n-2}\}$. Since V is smooth, these polynomials do not vanish simultaneously on a divisor, hence $D_{H^0(V,\Omega^1_V(2))} = 0$, and we are done. □

3.4 On Griffiths's proof of infinitesimal Torelli Theorem

In this section we will prove Theorem [C] of the Introduction.

It is well known by [9] that the deformation ξ is trivial if and only if R lies in the Jacobian ideal $\mathcal{J}$ of the variety V. The following lemma gives a translation of this condition in the setting of adjoint forms.

Lemma 3.6 *R is in the Jacobian ideal $\mathcal{J}$ if and only if* $\Omega \in \mathrm{Im}\,(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1)))$ *for the generic* Ω.

Proof If $\Omega \in \mathrm{Im}\,(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1)))$, then by the Adjoint Theorem, $\xi_{D_W} = 0$. Since $D_W = 0$, the deformation is trivial, hence R lies in the Jacobian ideal.

Viceversa if $R \in \mathcal{J}$, the deformation is trivial and by theorem (2.9), we have that $\Omega \in \mathrm{Im}\,(H^0(V, \mathcal{O}_V(2)) \otimes \lambda^n W \to H^0(V, \mathcal{O}_V(n+d-1)))$. □

Our theory gives another characterization for $[R] \in (\mathbb{C}[\xi_0, \dots, \xi_n]/\mathcal{J})_d \simeq H^1(V, \Theta_V)$ to be trivial.

Proposition 3.7 *Assume that* $\deg R = d > 3$. *Then* R *is in the Jacobian ideal* $\mathcal{J}$ *if and only if* $RP \in \mathcal{J}$ *for every polynomial* $P \in H^0(V, \mathcal{O}_V(n-1))$ *corresponding to a generalized adjoint* $\widetilde{\Omega} \in H^0(V, \Omega^n_{\mathbb{P}^n|V}(2n))$.

Proof One implication is trivial.

To prove the other one the idea is to show that every monomial of $H^0(V, \mathcal{O}_V(n-1))$ corresponds to a suitable generalized adjoint. Hence, if $RP \in \mathcal{J}$ for every polynomial $P \in H^0(V, \mathcal{O}_V(n-1))$ corresponding to a generalized adjoint, we have that $R \cdot H^0(V, \mathcal{O}_V(n-1)) \subset \mathcal{J}$ and we are done by Macaulay Theorem (c.f. [16] Theorem 6.19 and Corollary 6.20).

We work by induction at the level of $\mathbb{P}^n$, since $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n-1)) \to H^0(V, \mathcal{O}_V(n-1))$ is surjective. The base of the induction is for $n = 2$. A simple computation shows that the map

$$\bigwedge^2 H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(2)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \tag{3.31}$$

is surjective because its image contains the canonical basis of degree one monomials.

For the general case we show that every monomial of degree $n-1$ is given by a generalized adjoint. Consider the natural homomorphism:

$$\bigwedge^n H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n-1)) \tag{3.32}$$

and take a monomial M with $\deg M = n-1$. There is a variable ξ_i which does not appear in M. We restrict to the hyperplane $\xi_i = 0$ and we use induction on $\frac{M}{\xi_j}$, where ξ_j appears in M. There exist $s_1, \dots, s_{n-1} \in H^0(\mathbb{P}^{n-1}, \Omega^1_{\mathbb{P}^{n-1}}(2))$ with $s_1 \wedge \dots \wedge s_{n-1}$ which corresponds to $\frac{M}{\xi_j}$, that is

$$s_1 \wedge \dots \wedge s_{n-1} = \frac{M\Psi'}{\xi_j \cdot Q^{n-1}} \tag{3.33}$$

where $\Psi' = \sum_{k=0,k\neq i}^n (-1)^k \xi_k (d\xi_0 \wedge \dots \wedge \hat{d\xi_i} \dots \wedge \hat{d\xi_k} \dots \wedge d\xi_n)$ gives a basis of $H^0(\mathbb{P}^{n-1}, \Omega^{n-1}_{\mathbb{P}^{n-1}}(n))$. It is easy to see that

$$s_1 \wedge \dots \wedge s_{n-1} \wedge \frac{(\xi_j d\xi_i - \xi_i d\xi_j)}{Q} = \frac{M\Psi}{Q^n}, \tag{3.34}$$

i.e. M corresponds to a generalized adjoint, which is exactly our thesis. □

From the previous results we deduce immediately Theorem [C] of the Introduction.

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