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Geometric Optimization Problems Likely Not Contained in $\mathbb{APX}$*

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Abstract. Maximizing geometric functionals such as the classical l_p-norms over polytopes plays an important role in many applications, hence it is desirable to know how efficiently the solutions can be computed or at least approximated.

While the maximum of the l_∞-norm over polytopes can be computed in polynomial time, for $2 \le p < \infty$ the l_p-norm-maxima cannot be computed in polynomial time within a factor of 1.090, unless $\mathbb{P} = \mathbb{NP}$. This result holds even if the polytopes are centrally symmetric parallelotopes.

QUADRATIC PROGRAMMING is a problem closely related to norm-maximization, in that in addition to a polytope $P \subset \mathbb{R}^n$, numbers c_{ij}, $1 \le i \le j \le n$, are given and the goal is to maximize $\sum_{1 \le i \le j \le n} c_{ij} x_i x_j$ over P. It is known that QUADRATIC PROGRAMMING does not admit polynomial-time approximation within a constant factor, unless $\mathbb{P} = \mathbb{NP}$.

Using the observation that the latter result remains true even if the existence of an integral optimal point is guaranteed, in this paper it is proved that analogous inapproximability results hold for computing the l_p-norm-maximum and various l_p-radii of centrally symmetric polytopes for $2 \le p < \infty$.

Introduction

Geometric functionals of polytopes play an important role in numerous applications in mathematical programming, operations research, statistics, physics, chemistry, and medicine, see [GK3]. Of particular interest are the different l_p-radii and the l_p-norm-maxima, where for any vector $x = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n$ the l_p-norms are defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |\xi_i|^p \right)^{1/p} \qquad \text{for} \quad 1 \le p < \infty \quad \text{and} \quad \|x\|_\infty = \max_{1 \le i \le n} |\xi_i|.$$

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Using linear programming, $\max_{x \in P} \|x\|_\infty$, where P denotes the n-dimensional polytope given as input, can be computed in polynomial time [GK2]. Furthermore, it is trivial that once the vertices of P are explicitly known the pth power (in order to avoid non-rational numbers if $p \in \mathbb{N}$) of the l_p-norm-maximum over P can be computed in polynomial time since the maximum of the convex objective function is attained at a vertex. However, any (rational) polytope admits as well as a presentation as the convex hull of its vertices ($\mathcal{V}$-presentation, see precise definition in Section 1) a description by a system of linear inequalities ($\mathcal{H}$-presentation), but in general the number of vertices can be exponential in the number of inequalities and vice versa. Hence, from an algorithmic point of view it makes a crucial difference whether the polytope given as input is presented by its vertices or by a linear inequality system. Indeed, computing the pth power of the maximum of the l_p-norm over $\mathcal{H}$-presented polytopes, $p \in \mathbb{N}$, turns out to be $\mathbb{NP}$-complete even for simple sorts of polytopes like centrally symmetric parallelotopes [BGKL]. Hence it is unlikely (meaning unless $\mathbb{P} = \mathbb{NP}$) that the optimum can be exactly computed in polynomial time.

Consequently, one might ask for the hardness of approximation. If it is unlikely that norm-maxima can be exactly computed in polynomial time, can we at least derive "good" polynomial-time approximation algorithms? Actually, using an algorithm for special quadratic programs derived in [Y], the problem of maximizing the (square of the) Euclidean norm over centrally symmetric parallelotopes has a $\frac{7}{3}$-approximation algorithm. On the negative side, unless $\mathbb{P} = \mathbb{NP}$, there is no polynomial-time 1.090-approximation algorithm, see [BGK].

The latter results are proved only, in the case of the upper bound, and even, in the case of the lower bound, for very simple sorts of polytopes and the question arises for the situation in which general $\mathcal{H}$-polytopes are considered.

The current best upper bounds that can be achieved in the realm of the *Algorithmic Theory of Convex Bodies* in which general bodies are presented by oracles are $O(n^{p/2}/(\log n)^{p-1})$ for $1 \le p \le 2$ and $O(n/\log n)$ for $2 < p < \infty$. In this model the latter results are asymptotically optimal and in the Euclidean case even hold for randomized algorithms [BGK$^+$1], [BGK$^+$2].

In this paper the gap between the upper and lower bounds for general polytopes is significantly reduced by proving the non-existence of an algorithm that approximates l_p-norm-maxima within any constant factor, unless a relation holds that is widely believed to be false, i.e., $\mathbb{P} = \mathbb{NP}$.

Going along the same lines and using polarity, analogous results can be proved for the computation of the four main radii of polytopes, i.e., circumradius, inradius, diameter, and width.

The remainder of this paper is divided into three sections.

In Section 1 we start with the exact definition of the geometric optimization problems we are concerned with and state the main results. In Section 2 the inapproximability result for l_p-norm-maximization is proved by a reduction from the problem QUADRATIC PROGRAMMING in that in addition to a not full-dimensional polytope $P \subset \mathbb{R}^n$, numbers c_{ij}, $1 \le i \le j \le n$, are given and the goal is to maximize $\sum_{1 \le i \le j \le n} c_{ij} x_i x_j$ over P. The results for l_p-radii then follow from additional geometric transformations and are presented in Section 3.

1. Definitions and Statement of Main Results

The underlying model of computation is the binary Turing machine model. A string $(n, m; v_1, \ldots, v_m)$ with $n, m \in \mathbb{N}$ and $v_1, \ldots, v_m \in \mathbb{Q}^n$ is called a *$\mathcal{V}$-polytope* in $\mathbb{R}^n$; it represents the geometric object $P = \mathrm{conv}\{v_1, \ldots, v_m\}$. A string $(n, m; A, b)$, where $n, m \in \mathbb{N}$, A is a rational $m \times n$ matrix, $b \in \mathbb{Q}^m$, and the set $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded, is called an *$\mathcal{H}$-polytope* in $\mathbb{R}^n$; it is again identified with the geometric object P. The *binary size* $\langle P \rangle$ of a $\mathcal{V}$- or an $\mathcal{H}$-polytope P is the number of binary digits needed to encode the data of the presentation.

For $1 \leq p \leq \infty$ the *l_p-norm-maximum* $\max_{x \in P} \|x\|_p$ over a polytope $P \subset \mathbb{R}^n$ is denoted by $N_p(P)$. Furthermore, $r_p(P)$ denotes its inradius, $R_p(P)$ its circumradius, $d_p(P)$ its diameter, and $w_p(P)$ the width of the polytope P, where all these functionals are defined as usual, see [BGKL].

The precise definition of the algorithmical problems we are interested in is as follows:

Let $\mathcal{W} \in \{\mathcal{H}, \mathcal{V}\}$, let $p \geq 1$ be a rational number or $p = \infty$, and let $\varphi_p \in \{r_p^p, R_p^p, w_p^p, d_p^p, N_p^p\}$, with the understanding that φ_∞^∞ means φ_∞. Then we are interested in the following problem:

$(\varphi_p, \mathcal{W})$-COMPUTATION. Given a $\mathcal{W}$-presented polytope P as input, compute $\varphi_p(P)$.

We need some more notation in order to formulate the results precisely.

Suppose that a non-negative measurement ψ of polytopes is of interest, and A is an algorithm that produces, for each $\mathcal{W}$-presented polytope P, a number $\alpha(P)$. If, for a function $f\colon \mathbb{N} \longrightarrow [1, \infty[$, the number $\alpha(P)$ is always such that

$$\max\left\{\frac{\alpha(P)}{\psi(P)}, \frac{\psi(P)}{\alpha(P)}\right\} \leq f(\langle P \rangle),$$

then A is called an *f-approximation algorithm for* $(\psi, \mathcal{W})$-COMPUTATION and we say that *A approximates ψ* for $\mathcal{W}$-polytopes *with a* (*worst-case*) *performance ratio* f. Here $0/0$ has to be set to 1. In general, the case $\psi(P) = 0$ might cause difficulties and has to be explicitly dealt with depending on the specific problem. However, the case $\psi(P) = 0$ does not cause any difficulty for the choices of ψ considered here, because for them it can easily be decided in polynomial time whether $\psi(P) = 0$ and hence any algorithm can be augmented by a polynomial-time procedure that computes $\psi(P)$ correctly in this case [GK2]. In what follows we always assume that $\psi(P) > 0$, or, equivalently, we restrict ourselves to full-dimensional polytopes as input for $(\varphi_p, \mathcal{W})$-COMPUTATION.

The above definition naturally extends to other optimization problems and also to various classes of approximation algorithms. The *polynomial-time approximation algorithms* are of particular interest, and they are the subject of this paper. We say that $(\varphi_p, \mathcal{W})$-COMPUTATION does not admit polynomial-time f-approximation if every polynomial-time approximation algorithm has a performance ratio greater than f, and admits polynomial-time f-approximation if there exists a polynomial-time approximation algorithm whose performance ratio is at most f.

Problems for which a polynomial-time approximation algorithm with constant f exists belong by definition to the class $\mathbb{APX}$.

Finally, by $\tilde{\mathbb{P}}$ we denote the class of decision problems that can be solved in quasi-polynomial time, i.e., the running-time of an associated algorithm can be bounded by $m^{\log^{\nu} m}$, where ν is a positive constant and m denotes the size of the input.

Now let p be any rational number with $2 \le p < \infty$ and define p' by the equation $1/p + 1/p' = 1$. In this paper the following results are proved.

Theorem 1.1. *Unless* $\mathbb{P} = \mathbb{NP}$, *the problems* (i) $(N_p^p, \mathcal{H})$-COMPUTATION, (ii) $(d_p^p, \mathcal{H})$-COMPUTATION, (iii) $(R_p^p, \mathcal{H})$-COMPUTATION, (iv) $(w_{p'}^{p'}, \mathcal{V})$-COMPUTATION, *and* (v) $(r_{p'}^{p'}, \mathcal{V})$-COMPUTATION *are not contained in the class* $\mathbb{APX}$.

These results even hold for polytopes that are centrally symmetric with respect to the origin.

By weakening the assumption $\mathbb{P} \ne \mathbb{NP}$ the bound for the performance ratio can be improved as follows:

Theorem 1.2. *Unless* $\mathbb{NP} \subseteq \tilde{\mathbb{P}}$, *there exists a positive constant* $\delta < 1$ *such that the problems* (i) $(N_p^p, \mathcal{H})$-COMPUTATION, (ii) $(d_p^p, \mathcal{H})$-COMPUTATION, *and* (iii) $(R_p^p, \mathcal{H})$-COMPUTATION *do not admit polynomial-time* $2^{\log^{\delta} n}$*-approximation*, *where n denotes the dimension of the polytope given as input*.

2. Reducing Restricted Quadratic Programming to Norm-Maximization

Consider the following problems.

QUADRATIC PROGRAMMING. Given an $\mathcal{H}$-presented polytope $P \subset \mathbb{R}^n$ and numbers c_{ij}, $1 \le i \le j \le n$, as input, compute

$$\max_{x \in P} \sum_{1 \le i \le j \le n} c_{ij}\, x_i\, x_j, \qquad \text{where} \quad x = (x_1, \dots, x_n)^T.$$

RESTRICTED QUADRATIC PROGRAMMING. Given non-negative integral numbers κ, λ, σ, τ and non-negative rational numbers $c_{k,l,s,t}$, $k \in K := \{1, \dots, \kappa\}$, $l \in L := \{1, \dots, \lambda\}$, $s \in S := \{1, \dots, \sigma\}$, and $t \in T := \{1, \dots, \tau\}$ as input, compute the maximum of

$$f_1(x, y) = \sum_{\substack{k \in K, l \in L \\ s \in S, t \in T}} c_{k,l,s,t}\, x_{k,l}\, y_{s,t},$$

$x = (x_{1,1}, \dots, x_{1,\lambda}, x_{2,1}, \dots, x_{\kappa,\lambda})$ and $y = (y_{1,1}, \dots, y_{1,\tau}, y_{2,1}, \dots, y_{\sigma,\tau})$, over the (not full-dimensional) polytope $P_1 \subset \mathbb{R}^{\kappa\lambda + \sigma\tau}$ described by the system

$$\sum_{k \in K} x_{k,l} = 1 \qquad \text{for} \quad l \in L,$$

$$\sum_{s \in S} y_{s,t} = 1 \qquad \text{for} \quad t \in T,$$

$$0 \le x_{k,l} \le 1 \qquad \text{for} \quad k \in K, \quad l \in L, \quad \text{and}$$
$$0 \le y_{s,t} \le 1 \qquad \text{for} \quad s \in S, \quad t \in T.$$

Of course, the second problem is a more restricted version of the first one, hence inapproximability results for the latter yield the same results for the more general problem.

It will be convenient to denote by ζ-RESTRICTED QUADRATIC PROGRAMMING the problem in that in addition a positive number ζ is fixed and only instances of RESTRICTED QUADRATIC PROGRAMMING are considered that satisfy $\kappa, \sigma \le \zeta$.

Feige and Lovász [FL] and independently Bellare and Rogaway [BR] related RESTRICTED QUADRATIC PROGRAMMING to two-prover one-round interactive proof systems and proved first inapproximability results. Together with subsequent work by Feige and Kilian on interactive proof systems [FK] we obtain the following propositions.

Proposition 2.1. *Unless* $\mathbb{P} = \mathbb{NP}$, RESTRICTED QUADRATIC PROGRAMMING *is not contained in the class* $\mathbb{APX}$.

In fact, we need a more quantitative version:

Proposition 2.2. *For any* $\alpha \ge 1$ *there exists a positive integer* $\zeta = \zeta(\alpha)$ *such that* ζ-RESTRICTED QUADRATIC PROGRAMMING *does not admit polynomial-time* α-*approximation*.

Proposition 2.3. *Unless* $\mathbb{NP} \subseteq \tilde{\mathbb{P}}$, *there exists a positive constant* $\delta < 1$ *such that* RESTRICTED QUADRATIC PROGRAMMING *does not admit polynomial-time* $2^{\log^{\delta} n}$-*approximation*, *where n denotes the dimension of the polytope given as input*.

It is important to note that the existence of an integral point that maximizes f_1 over P_1 is always guaranteed. This property is crucial for the following transformation to l_p-norm-maximization, $2 \le p < \infty$, that is divided into three auxiliary steps. In each step we obtain a new polytope P_i and we denote by f_i^* the maximum of the objective function f_i that has to be maximized over P_i, $2 \le i \le 4$.

The first step of our reduction consists of replacing the equalities in the presentation of P_1 by proper inequalities, i.e., we define the polytope P_2 by the system

$$\sum_{k \in K} x_{k,l} \le 1 \qquad \text{for each} \quad l \in L,$$
$$\sum_{s \in S} y_{s,t} \le 1 \qquad \text{for each} \quad t \in T,$$
$$0 \le x_{k,l} \le 1 \qquad \text{for} \quad k \in K, \quad l \in L, \quad \text{and}$$
$$0 \le y_{s,t} \le 1 \qquad \text{for} \quad s \in S, \quad t \in T,$$

of $\lambda + \tau + 2(\kappa\lambda + \sigma\tau)$ inequalities in $\kappa\lambda + \sigma\tau$ variables.

Choosing $f_2 = f_1$ we obtain the following lemma.

Lemma 2.4. *P_2 is full-dimensional and* $f_1^* = f_2^*$.

Proof. Just note that the point $1/(1+\max\{\kappa,\sigma\})(1,\dots,1)^T$ is an interior point of P_2 and that all coefficients of the objective function are non-negative. □

Next consider the polytope P_3 that is defined by the following system:

$$\begin{aligned}
\sum_{k\in K} z_{k,l,s_k,t_k} &\le 1 \qquad \text{for} \quad l\in L, \quad s_k\in S, \quad t_k\in T,\\
\sum_{s\in S} z_{k_s,l_s,s,t} &\le 1 \qquad \text{for} \quad t\in T, \quad k_s\in K, \quad l_s\in L, \quad \text{and}\\
0\le z_{k,l,s,t,} &\le 1 \qquad \text{for} \quad k\in K, \quad l\in L, \quad s\in S, \quad t\in T,
\end{aligned}$$

of $\lambda(\sigma\tau)^\kappa + \tau(\kappa\lambda)^\sigma + 2(\kappa\lambda\sigma\tau)$ inequalities in $\kappa\lambda\sigma\tau$ variables, with the associated objective function

$$f_3(z) = \sum_{\substack{k\in K,l\in L\\ s\in S,t\in T}} c_{k,l,s,t}\, z^p_{k,l,s,t},$$

where $z=(z_{1,1,1,1},\dots,z_{\kappa,\lambda,\sigma,\tau})^T$.

Lemma 2.5. *P_3 is a full-dimensional polytope and $f_2^* = f_3^*$.*

Proof. Since the first part of the lemma is trivial we prove $f_2^*=f_3^*$.

In order to show $f_2^*\ge f_3^*$ take any $z\in P_3$ and define

$$x=(x_{1,1},\dots,x_{\kappa,\lambda})^T \qquad \text{by} \quad x_{k,l} = \max_{s\in S,t\in T} z_{k,l,s,t} \quad \text{for} \quad k\in K, \quad l\in L,$$

and

$$y=(y_{1,1},\dots,y_{\sigma,\tau})^T \qquad \text{by } y_{s,t} = \max_{k\in K,l\in L} z_{k,l,s,t} \quad \text{for} \quad s\in S, \quad t\in T.$$

We obtain, using $z\in P_3$,

$$\begin{aligned}
\sum_{k\in K} x_{k,l} &= \sum_{k\in K} z_{k,l,s_k,t_k} \le 1 \qquad \text{for} \quad l\in L,\\
\sum_{s\in S} y_{s,t} &= \sum_{k\in K} z_{k_s,l_s,s,t} \le 1 \qquad \text{for} \quad t\in T,
\end{aligned}$$

where s_k, t_k, k_s, and l_s can be chosen properly, and $0\le x_{k,l}, y_{s,t}\le 1$ for $k\in K$, $l\in L$, $s\in S$, and $t\in T$. Hence $(x^T,y^T)^T\in P_2$ and, furthermore, since $z_{k,l,s,t}\le x_{k,l}, y_{s,t}$ for $k\in K$, $l\in L$, $s\in S$, and $t\in T$,

$$\sum_{\substack{k\in K,l\in L\\ s\in S,t\in T}} c_{k,l,s,t}\, x_{k,l}\, y_{s,t} \ge \sum_{\substack{k\in K,l\in L\\ s\in S,t\in T}} c_{k,l,s,t}\, z^2_{k,l,s,t} \ge \sum_{\substack{k\in K,l\in L\\ s\in S,t\in T}} c_{k,l,s,t}\, z^p_{k,l,s,t}.$$

Now take any $(x^T,y^T)^T\in P_2$ and define z by $z_{k,l,s,t,} = \min\{x_{k,l}, y_{s,t}\}$ for $k\in K$, $l\in L$, $s\in S$, and $t\in T$. We obtain, using $(x^T,y^T)^T\in P_2$,

$$\begin{aligned}
\sum_{k\in K} z_{k,l,s_k,t_k} &\le \sum_{k\in K} x_{k,l} \le 1 \qquad \text{for} \quad l\in L, \quad s_k\in S, \quad t_k\in T,\\
\sum_{k\in K} z_{k_s,l_s,s,t} &\le \sum_{s\in S} y_{s,t} \le 1 \qquad \text{for} \quad t\in T, \quad k_s\in K, \quad l_s\in L,
\end{aligned}$$

and $0 \leq z_{k,l,s,t} \leq 1$ for $k \in K$, $l \in L$, $s \in S$, and $t \in T$. It follows that $z \in P_3$. Furthermore, for any integral point $(x^T, y^T)^T \in P_2$ the associated point $z \in P_3$ is also integral and $z_{k,l,s,t} = 1$ iff $x_{k,l} = 1$ and $y_{s,t} = 1$. This implies $z_{k,l,s,t}^p = x_{k,l} y_{s,t}$ for $k \in K$, $l \in L$, $s \in S$, and $t \in T$ and by the existence of an integral point that maximizes f_2 over P_2 we conclude $f_2^* \leq f_3^*$. □

Next, note that for our purpose we may assume without loss of generality that $c_{k,l,s,t} > 0$ for $k \in K$, $l \in L$, $s \in S$, and $t \in T$. (Otherwise proper scaling is necessary.) Hence we may replace the variables $z_{k,l,s,t}$ by $c_{k,l,s,t}^{-1/p} \hat{z}_{k,l,s,t}$, $k \in K$, $l \in L$, $s \in S$, $t \in T$, and obtain the polytope $\hat{P}_4$ described by the system

$$\begin{aligned}
\sum_{k \in K} c_{k,l,s_k,t_k}^{-1/p} \hat{z}_{k,l,s_k,t_k} &\leq 1 \qquad \text{for} \quad l \in L, \quad s_k \in S, \quad t_k \in T, \\
\sum_{s \in S} c_{k_s,l_s,s,t}^{-1/p} \hat{z}_{k_s,l_s,s,t} &\leq 1 \qquad \text{for} \quad t \in T, \quad k_s \in K, \quad l_s \in L, \quad \text{and} \\
0 \leq c_{k,l,s,t}^{-1/p} \hat{z}_{k,l,s,t}, &\leq 1 \qquad \text{for} \quad k \in K, \quad l \in L, \quad s \in S, \quad t \in T,
\end{aligned}$$

and we consider the objective function $f_4(\hat{z}) = \|\hat{z}\|_p^p$, where $\hat{z}$ is canonically defined.

Trivially, $f_3^* = f_4^*$, but of course, in general, an $\mathcal{H}$-presentation of $\hat{P}_4$ does not exist. However, using standard rounding techniques we obtain the following lemma.

Lemma 2.6. *Let $\omega > 1$ be a rational number. Then there exists a polynomial-time algorithm that given P_3 as input produces an $\mathcal{H}$-polytope P_4 such that for $f_4 = \|\cdot\|_p^p$,*

$$\frac{1}{\omega} f_3^* \leq f_4^* \leq \omega f_3^*.$$

Theorem 2.7. *Let A be a polynomial-time approximation-algorithm for $(N_p^p, \mathcal{H})$-*COMPUTATION *with constant performance ratio ρ. Then there exists for any $\omega > 1$ a polynomial-time approximation algorithm A' for ζ-*RESTRICTED QUADRATIC PROGRAMMING *with performance ratio $\omega\rho$, where $\zeta = \zeta(\omega\rho)$ as in Proposition* 2.2.

Proof. Consider an instance I of ζ-RESTRICTED QUADRATIC PROGRAMMING. First, reduce I to an instance of $(N_p^p, \mathcal{H})$-COMPUTATION, using ω as a parameter in the last step (see Lemma 2.6). Since the upper bound ζ for κ and σ is independent of the input this can be done in polynomial time. Now, use A to compute $\tilde{f}_4^*$ with $1/\rho \tilde{f}_4^* \leq f_4^* \leq \rho \tilde{f}_4^*$ and choose $\tilde{f}_1^* := \tilde{f}_4^*$ to approximate f_1^*. Using the lemmas proved in this section we conclude

$$\frac{1}{\omega\rho} \leq \frac{1}{\omega} \frac{f_4^*}{\tilde{f}_4^*} \leq \frac{\omega f_3^*}{\omega \tilde{f}_4^*} = \frac{f_1^*}{\tilde{f}_1^*} = \frac{f_3^*}{\tilde{f}_4^*} \leq \frac{\omega f_4^*}{\tilde{f}_4^*} \leq \omega\rho. \qquad \square$$

Hence, combining Theorem 2.7 with Proposition 2.2 yields the first part of Theorem 1.1.

Using a slightly different transformation an analogous result for RESTRICTED QUADRATIC PROGRAMMING can be proved.

Theorem 2.8. *Let A be a polynomial-time approximation-algorithm for $(N_p^p, \mathcal{H})$-*COMPUTATION *with performance ratio ρ.*

(i) *Then there exists for any $\omega > 1$ a polynomial-time approximation algorithm A' for* RESTRICTED QUADRATIC PROGRAMMING *with performance ratio $\omega\rho$.*
(ii) *In addition, let n (respectively $\hat{n}$) denote the dimension of the polytope given as input for $(N_p^p, \mathcal{H})$*-COMPUTATION (*respectively* RESTRICTED QUADRATIC PROGRAMMING), *and let the performance ratio ρ depend on n, i.e., $\rho = \rho(n)$. Then A' has performance ratio $\rho(\hat{n}^2)$.*

Together with Proposition 2.3 this yields the first part of Theorem 1.2. For the second and third parts we use that for polytopes containing the origin norm-maximum, diameter and circumradius differ only by a constant independent of the dimension of the polytope.

However, polytopes obtained by the transformation described above can be easily transformed into centrally symmetric ones, a property crucial for the proof of the fourth and fifth parts of Theorem 1.1.

3. Computation of l_p-Radii of Polytopes

For a convex body K that is centrally symmetric with respect to the origin, i.e., K is a full-dimensional, compact, and convex set with $K = -K$, the norm-maximum is closely related to the diameter, circumradius, width, and inradius [GK1]. For any p with $1 \le p \le \infty$ we have the following:

Proposition 3.1. $w_p(K) \ge 2r_p(K)$ *and* $d_p(K) \le 2R_p(K)$, *with equality when K is symmetric.*

Proposition 3.2. *If $K = -K$, then $r_p(K)\mathbb{B}_p \subseteq K \subseteq R_p(K)\mathbb{B}_p$, where $\mathbb{B}_p$ is the l_p-unit ball.*

Proposition 3.3. *If $K = -K$, then $R_p(K)r_{p'}(K^\circ) = 1$, where K° denotes the polar of K, i.e., the set of all (linear) functionals f in the conjugate space (here $l_{p'}$) such that $f(x) \le 1$ for all $x \in K$.*

Hence, regarding the results for norm-maximization derived in the previous section the question arises whether similar inapproximability results hold for the computation of the other functionals.

This question will be answered in the positive by transforming $\tilde{P}_4$ into a centrally symmetric polytope $\tilde{P}_4^c$ that is defined by the following system:

$$\sum_{k\in K} \varepsilon_k c_{k,l,s_k,t_k}^{-1/p} \hat{z}_{k,l,s_k,t_k} \le 1 \qquad \text{for} \quad l \in L, \quad s_k \in S, \quad t_k \in T, \quad \varepsilon_k \in \{-1, 1\},$$

$$\sum_{s\in S} \varepsilon_s c_{k_s,l_s,s,t}^{-1/p} \hat{z}_{k_s,l_s,s,t} \le 1 \qquad \text{for} \quad t \in T, \quad k_s \in K, \quad l_s \in L, \quad \varepsilon_s \in \{-1, 1\},$$

$$-1 \le c_{k,l,s,t}^{-1/p} \hat{z}_{k,l,s,t,} \le 1 \qquad \text{for} \quad k \in K, \quad l \in L, \quad s \in S, \quad t \in T,$$

of $\lambda(2\sigma\tau)^\kappa + \tau(2\kappa\lambda)^\sigma + 2(\kappa\lambda\sigma\tau)$ inequalities in $\kappa\lambda\sigma\tau$ variables.

It is easy to see that the optimum of f_4 over $\tilde{P}_4$ equals the optimum of f_4 over P_4^c. Hence, using Propositions 3.1 and 3.2, we obtain the following for each $p \geq 2$.

Theorem 3.4. *Let A be a polynomial-time approximation-algorithm for* $(d_p^p, \mathcal{H})$-COMPUTATION ($(R_p^p, \mathcal{H})$-COMPUTATION) *with constant performance ratio ρ. Then there exists for any $\omega > 1$ a polynomial-time approximation algorithm A' for* ζ-RESTRICTED QUADRATIC PROGRAMMING *with performance ratio $\omega\rho$, where $\zeta = \zeta(\omega\rho)$ as in Proposition* 2.2.

Finally, Proposition 3.3 yields

Corollary 3.5. *Let A be a polynomial-time approximation-algorithm for* $(w_{p'}^{p'}, \mathcal{V})$-COMPUTATION ($(r_{p'}^{p'}, \mathcal{V})$-COMPUTATION), *with performance ratio ρ. Then there exists for any $\omega > 1$ a polynomial-time approximation algorithm A' for* ζ-RESTRICTED QUADRATIC PROGRAMMING *with performance ratio $\omega\rho$, where $\zeta = \zeta(\omega\rho)$ as in Proposition* 2.2.

Together with Proposition 2.2 these results prove the remaining parts of Theorem 1.1.

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