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On regularization of plurisubharmonic functions near boundary points

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Abstract We prove in an elementary way that for a Lipschitz domain $D \subset \mathbb{C}^n$, all plurisubharmonic functions on D can be regularized near any boundary point.

Keywords Plurisubharmonic function · Regularisation

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1 Introduction

Let $D \subset \mathbb{C}^n$. Using a local convolution, for any plurisubharmonic function u on D, one can find a sequence u_k of smooth plurisubharmonic functions, which decreases to u on compact subsets of D. The purpose of this note is to show that (for regular enough D) it is possible to choose such a sequence near any point on the boundary.

A domain $D \subset \mathbb{C}^n$ will be called an $\mathcal{S}$-domain if for any plurisubharmonic function u on D, one can find a sequence u_k of smooth plurisubharmonic functions on D, which decreases to u.[1] Note that pseudoconvex domains, tube domains and Riendhard domains are $\mathcal{S}$-domains (see [3]).

Theorem 1 *Let $D \subset \mathbb{C}^n$ be a domain with Lipschitz boundary. Then for any boundary point P there is a neighbourhood U of P such that $D \cap U$ is an $\mathcal{S}$-domain with Lipschitz boundary.*

In Sect. 3, by a slight modification of an example from [2], we show the necessity of the assumption on the boundary.

[1] Of course this notion does not depend on coordinates so we can define it on manifolds (for example compact manifolds are $\mathcal{S}$-domains). Theorem 1 can be also formulated on manifolds.

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Because of the fact that not every smooth domain is an $\mathcal{S}$-domain (see [1]) we have the following (surprising for the author) corollary:

Corollary 2 *Being an $\mathcal{S}$-domain is not a local property of the boundary.*

2 Proof

We need the following lemma:

Lemma 3 *Let $D \subset \mathbb{R}^m$ be an open set. Let u be a subharmonic function on D and let $P \in D$. Let $a, b, R, C > 0$, $B = \{x \in \mathbb{R}^m : |x - P| < R\}$, $K = \{(x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m : -a < x_m < -b|x'|\}$ and $B + K = \{x + y : x \in B, y \in K\} \subset D$. Assume that for any $x \in B$ and $y \in K$*

$$u(x + y) \leq u(x) + \delta(|y|),$$

where $\delta : (0, +\infty) \to (0, +\infty)$ is increasing and such that $\lim_{t \to 0^+} \delta(t) = 0$. Then u is continuous at P.

Proof Let (x_n) be any sequence in D which converges to P. Let $S_n = \{x \in \mathbb{R} : |x - P| = 2|x_n - P|\}$, $A_n = S_n \cap (\{x_n\} + K)$ and $B_n = S_n \backslash A_n$. For n large enough $x_n \in B$ and $|x_n - P| \leq \frac{a}{3}$. Hence there is a constant $\alpha > 0$ (which depends only on b) such that $\alpha_n = \frac{\sigma(A_n)}{\sigma(S_n)} \geq \alpha$ where σ is the standard measure on a sphere. Let $M_n = sup_{S_n} u$. We can estimate using the assumptions:

$$\begin{aligned} u(P) &\leq \sigma(S_n)^{-1} \int_{A_n} u d\sigma + \sigma(S_n)^{-1} \int_{B_n} u d\sigma \\ &\leq \alpha_n(u(x_n) + \delta(3|x_n - P|)) + (1 - \alpha_n)M_n, \end{aligned}$$

hence

$$u(x_n) \geq u(P) + \frac{1 - \alpha_n}{\alpha_n}(u(P) - M_n) - \delta(3|x_n - P|).$$

Letting n to ∞ we get

$$\underline{\lim}_{n \to \infty} u(x_n) \geq u(P).$$

Since u is upper semicontinuous the proof is completed. □

The function $P_D f := \sup\{u \in \mathcal{PSH}(D) : u \leq f\}$, where $D \subset \mathbb{C}^n$ and f is a (real) function on D, is called a plurisubharmonic envelope of f.

Proof of Theorem 1 We use the following notation $\mathbb{C}^n \ni z = (a, x) \in (\mathbb{C}^{n-1} \times \mathbb{R}) \times \mathbb{R}$. We put $B = \{a \in \mathbb{C}^{n-1} \times \mathbb{R} : |a| < 1\}$. After an affine change of coordinates we can assume that there is a constant $C > 0$ and a function $F : B \to (3C, 4C)$ such that:

(i) $F(a) - F(b) \leq C|a - b|$ for $a, b \in B$,
(ii) $\partial D \cap B \times [-5C, 5C] = \{(a, F(a)) : a \in B\}$ and $P = (0, F(0))$,
(iii) $0 \in D$.

Let $U = \{(a, x) \in B \times \mathbb{R} : |a|^2 + \left(\frac{x}{5C}\right)^2 < 1\}$. We will show that $\Omega = D \cap U$ is an $\mathcal{S}$-domain. Observe that

$$\partial\Omega \cap (B \times [0, 5C]) = \{(a, \hat{F}(a)) : a \in B\},$$

where $\hat{F}(a) = \min\{F(a), 5C\sqrt{1-|a|^2}\}$. By elementary calculations we get

$$\{\hat{F} = F\} \subset B' = \left\{a \in \mathbb{C}^{n-1} \times \mathbb{R} : |a| < \frac{4}{5}\right\} \tag{1}$$

and

$$\hat{F}(a) - \hat{F}(b) \leq C'|a-b| \quad \text{for } a, b \in B', \tag{2}$$

where $C' = \frac{20}{3}C < 7C$. For $0 < \varepsilon < \sup_{B'} F - 3C$ let

$$K = K(\varepsilon) = \{(a, x) : -\varepsilon < x < -7C|a|\}$$

and let

$$\Omega_k = \Omega_k(\varepsilon) = \{z \in \Omega : z + kw \in \Omega \text{ for any } w \in \bar{K}\}, \text{ for } k = 1, 2.$$

By (1) and (2), we have $\partial\Omega_2 \cap \partial D = \partial\Omega \cap \partial D$. This gives us

$$\partial\Omega_2 = (\partial\Omega_2 \cap \partial D) \cup (\partial\Omega_2 \cap \partial U) \cup (\partial\Omega_2 \cap \Omega_1)\,.$$

Therefore by (2) it is clear that

$$z + w \in \Omega$$

for $z \in \partial\Omega_2 \cap \partial D$ and $w \in \bar{K}$. The same inequality holds on $\partial\Omega_2 \cap \partial U$ because of the convexity of U. Thus we get

$$L = L(\varepsilon) := \{z \in \bar{\Omega}_2 : \operatorname{dist}(z, \partial\Omega) \geq \operatorname{dist}(z + w, \partial\Omega) \text{ for some } w \in \bar{K}\} \Subset \Omega_1.$$

Let $d = -\log(\operatorname{dist}(\cdot, \partial\Omega))$ and $d' = -\log(\operatorname{dist}(\cdot, \partial U))$. We only know that the second function is plurisubharmonic, but by the construction of Ω and Ω_2 (decreasing ε if necessary) we have $d = d'$ on $\Omega \backslash \Omega_2$, hence on this set, the function d is plurisubharmonic too.

Let $u \in \mathcal{PSH}(\Omega)$ and let ϕ_k be a sequence of continuous functions on Ω which decreases to u. We can choose an increasing convex function $p : \mathbb{R} \to \mathbb{R}$ such that for a function $\rho = p \circ d$ we have $\lim_{z \to \partial\Omega} \rho - \phi_1 = +\infty$ (see claim 3.5 in [5]). Put $\tilde{\phi}_k = \max\{\phi_k, \rho - k\}$. Observe that functions $\hat{\phi}_k = P_\Omega \tilde{\phi}_k$ are plurisubharmonic and they decrease to u.

Fix k. Because $\tilde{\phi}_k = \rho - k$ outside of a compact set, for $\varepsilon > 0$ small enough, we have

$$cl_\Omega(\Omega \backslash \Omega_2) \subset S,$$

where $S = S(\varepsilon) = int\{\tilde{\phi}_k = \rho - k\}$.
The function $\rho' := p \circ d'$ is plurisubharmonic and $\rho' \leq \tilde{\phi}_k$. Thus on Ω_2 we have

$$\tilde{\phi}_k \geq P_{\Omega_2}\tilde{\phi}_k \geq \hat{\phi}_k \geq \rho' - k$$

and therefore the function v given by

$$v(z) = \begin{cases} P_{\Omega_2}\tilde{\phi}_k(z) & \text{for } \; z \in \Omega_2 \\ \rho(z)' - k (= \tilde{\phi}_k) & \text{on } \; \Omega \backslash \Omega_2, \end{cases}$$

is plurisubharmonic on Ω and smaller than $\tilde{\phi}_k$. Thus $P_{\Omega_2}\tilde{\phi}_k = \hat{\phi}_k|_{\Omega_2}$. Let

$$L' = L'(\varepsilon) := \{z \in \bar{\Omega}_2 : \rho(z) \geq \sup_{\Omega \backslash S} \phi_k\} \Subset \Omega_1,$$

then we have

$$N = N(\varepsilon) := (L \cup L') + \bar{K} \Subset \Omega.$$

Observe that for $z \in \Omega_2$ and $w \in K$ we have $\tilde{\phi}_k(z) \geq \tilde{\phi}_k(z+w) - \omega(|w|)$, where ω is the modulus of continuity of the function $\tilde{\phi}_k|_N$. Therefore, $\hat{\phi}_k(z) \geq \hat{\phi}_k(z+w) - \omega(|w|)$. By Lemma 3 the function $\hat{\phi}_k|_{\Omega_2}$ is continuous.

Because any $z \in \Omega$ is in $\Omega_2(\varepsilon)$ for some ε as above, we obtain that the function $\hat{\phi}_k$ is continuous on Ω. Using the Richberg theorem we can modify the sequence $\hat{\phi}_k$ to a sequence u_k of smooth plurisubharmonic functions which decreases to u. □

The approximation by continuous functions, can be proved in the same way in a much more general situation.

Theorem 4 *Let $\boldsymbol{F}$ be a constant coefficient subequation such that all $\boldsymbol{F}$-subharmonic functions are subharmonic and all convex functions are $\boldsymbol{F}$-subharmonic. Let $D \subset \mathbb{R}^n$ be a domain with $Lipschitz$ boundary. Then for any point $P \in \bar{D}$ there is a neighbourhood U of P such that for any function $u \in \boldsymbol{F}(D \cap U)$ there is a sequence $(u_k) \subset \boldsymbol{F}(D \cap U)$ of continuous functions decreasing to u.*

Here we use terminology from [4].[2]

3 Example

Similarly as in Lecture 14 in [2] we construct a domain $\Omega \subset \mathbb{C}^n$ and a plurisubharmonic function u on Ω which can not be regularized. Let $A = \{\frac{1}{k} : k \in \mathbb{N}\}$ and a sequence $(x_k) \subset (0,1)\setminus A$ is such that its limit set is equal $\bar{A}$. Put

$$\lambda(z) = \sum_{k=1}^{\infty} c_k \log|z - x_k| \quad \text{for } z \in \mathbb{C},$$

where c_k is a such sequence of numbers rapidly decreasing to 0, such that

(i) λ is a subharmonic function on $\mathbb{C}$ and
(ii) $\lambda|_A \geq -\frac{1}{2}$.

For $k \in \mathbb{N}$ let D_k be a disc with center x_k such that $\lambda|_{D_k} < -1$. Now, we can define

$$\Omega = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'|^2 + |z_n - 1|^2 < 1\}\setminus K,$$

where

$$K = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| = |z_n| \text{ and } z_n \notin \cup_{k=1}^{\infty} D_k\},$$

and

$$u(z', z_n) = \begin{cases} -1 & \text{for} \quad z \in \Omega \cap D \\ \max\{\lambda(z_n), -1\} & \text{on} \quad \Omega\setminus D, \end{cases}$$

where $D = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| > |z_n|\}$.

Let U be any neighbourhood of 0. We can choose numbers $k \in \mathbb{N}$, $0 < r < \frac{1}{k}$ such that

$$\Omega_U := \left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| \leq \frac{2}{k}, |z_n - \frac{1}{k}| \leq r\right\}\setminus K \subset \Omega \cap U.$$

[2] See also Theorem 2.6 there for elementary properties of **F**-subharmonic functions needed in the proof. For the result of local continuous approximation of F-subharmonic functions see [5]

Let y_p be a subsequence of x_p such that $\lim_{p\to\infty} y_p = \frac{1}{k}$ and for all p we have $|y_p - \frac{1}{k}| < r$. Now, we can repeat the argument from [2]. If u_q is a sequence of smooth plurisubharmonic functions decreasing to u on Ω_U, then for q sufficiently large $u_q \leq -\frac{3}{4}$ on the set

$$\left\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| = \frac{2}{k}, |z_n - \frac{1}{k}| \leq r\right\} \subset \partial\Omega_U.$$

By the maximum principle (on sets $\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z'| \leq \frac{2}{k}, z_n = y_p\} \subset \mathbb{C}^{n-1} \times \{y_p\}$) we also have $u_q(0, y_p) \leq -\frac{3}{4}$ and by continuity of u_q we get $u_q(0, \frac{1}{k}) \leq -\frac{3}{4} < u(0, \frac{1}{k})$. This is a contradiction.

Note that we have not only just proved that for the above Ω Theorem 1 does not hold but we also have the following stronger result:

Proposition 5 *Let Ω and u be as above. Then the function u can not be smoothed on $U \cap \Omega$ for any neighbourhood U of $0 \in \partial\Omega$.*

4 Questions

In this last section we state some open questions related to the content of the note.

1. Is it possible to characterize $\mathcal{S}$-domains? In view of Corollary 2, even in the class of smooth domains, it seems to be a challenging problem.
2. Let D be an $\mathcal{S}$-domain and let f be a continuous function which is bounded from below. Is the plurisubharmonic envelope of f continuous? Assume in addition that f is smooth. What is the optimal regularity of $P_D f$? The author does not know answers even in the case of the ball in $\mathbb{C}^n$. Note that if D is not an $\mathcal{S}$-domain, then there is a smooth function f bounded from below such that $P_D f$ is discontinuous.
3. What is the optimal assumption about the regularity of the boundary of D in the Theorem 1? Is it enough to assume that $D = int\bar{D}$?
4. Let M be a real (smooth or Lipschitz) hypersurface in $\mathbb{C}^n$. Is it true that for any $P \in M$ there exists a smooth pseudoconvex neighbourhood $U \subset B$ such that M divides U into two $\mathcal{S}$-domains?

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