LEVEL SET METHODS FOR A PARAMETER IDENTIFICATION PROBLEM

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1. Introduction

Consider the partial differential equation

$$\begin{cases} -\nabla \cdot (q(x)\nabla u) &= f \quad \text{in } \Omega \subset R^d, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

We want to use observations of the solution u to recover the coefficient $q(x)$. We shall especially treat the case that $q(x)$ has discontinuities and is piecewise constant.

In this work, we shall combine the ideas used in [5] and [2] to use level set methods to estimate the coefficient $q(x)$. The level set method was first proposed in Osher and Sethian [8]. This method associate a two-dimensional closed curve with a two-dimensional function. Extensions to higher dimensions are also easy, see Ambrosio and Soner [1]. The

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: 10.1007/978-0-387-35690-7_44
V. Barbu et al. (eds.), *Analysis and Optimization of Differential Systems*

advantage of using level set method is that it gives a better tool for evolving curves that may disappear, merge with each other, or pinch off with each other.

The level set method has been used for some inverse problems in [4, 10], etc. The work of [5] seems to be the first one to apply the level set idea to estimate the coefficient $q(x)$ from the equation (1). The work of [5] only works when the coefficient $q(x)$ takes two constants values, i.e. they can only handle one level set function. Several approaches have been proposed to use multiple level set functions (not for inverse problems, but for fluid and image problems) [2, 13]. The approach of [2] is easy to implement and has been well tested for image segmentation problems. In this work, we are trying to use the idea of [2] for the parameter estimation problem. See also [11, 12] for some related works in using Heaviside functions to identify shapes and boundaries.

2. Level set methods

Here, we state some of the details of the level set idea. Let Γ be a closed curve in Ω. Associated with Γ, we define ϕ as a signed distance function by

$$\phi(x) = \begin{cases} \text{distance}(x, \Gamma), & x \in \text{interior of } \Gamma \\ -\text{distance}(x, \Gamma), & x \in \text{exterior of } \Gamma, \end{cases}$$

In many applications, the movement of the curve Γ can be described by a partial differential equation of the function ϕ. The function ϕ is called a level set function for Γ. In fact, ϕ is the unique viscosity solution null on Γ for the following partial differential equation

$$|\nabla \phi| = 1, \quad \text{in } \Omega. \tag{2}$$

In this work, we shall use the level set method to identify the coefficient q which is assumed to be piecewise constant. First look at a simple case, i.e. assume that q has a constant value q_1 inside a closed curve Γ and is another constant q_2 outside the curve Γ. Utilizing the Heaviside function $H(\phi)$, which is equal to 1 for positive ϕ and 0 elsewhere, it is easy to see that q can be represented as

$$q = q_1 H(\phi) + q_2 (1 - H(\phi)). \tag{3}$$

In order to identify the coefficient q, we just need to identify a level set function ϕ and the piecewise constant values q_i.

If a function has many pieces, then we need to use multiple level set functions. This idea was introduced in Chan and Vese [2]. Assume that we have two closed curves Γ_1 and Γ_2, and we associate the two level set functions $\phi_j, j = 1,2$ with these curves. Then the domain Ω is divided into the four pars $\Omega_{++} = \{x\in\Omega,\ \phi_1 > 0,\ \phi_2 > 0\}$, $\Omega_{+-} = \{x\in\Omega,\ \phi_1 > 0,\ \phi_2 \leq 0\}$, $\Omega_{-+} = \{x\in\Omega,\ \phi_1 \leq 0,\ \phi_2 > 0\}$, $\Omega_{--} = \{x\in\Omega,\ \phi_1 \leq 0,\ \phi_2 \leq 0\}$.

Allowing some of the subdomains defined above to be empty, we can easily handle the case that the zero level set curves could merge, split or disappear. Using the Heaviside function again, we can express q with possibly up to four pieces with constant values as

$$\begin{aligned} q &= q_1 H(\phi_1)H(\phi_2) + q_2 H(\phi_1)(1 - H(\phi_2)) + \\ &\quad + q_3(1 - H(\phi_1))H(\phi_2) + q_4(1 - H(\phi_1))(1 - H(\phi_2)). \end{aligned} \tag{4}$$

By generalizing, we see that n level set functions give the possibility of 2^n regions. In that case, q would look like

$$\begin{aligned} q &= q_1 H(\phi_1) H(\phi_2) \cdots H(\phi_n) + \\ &\quad + q_2 (1 - H(\phi_1)) H(\phi_2) \cdots H(\phi_n) + \\ &\quad \vdots \\ &\quad + q_{2^n} (1 - H(\phi_1)) (1 - H(\phi_2)) \cdots (1 - H(\phi_n)) . \end{aligned} \tag{5}$$

Even if we need less than 2^n distinct regions, we can still use n level set functions since some subdomains may be empty. In using such a representation, we need to determine the maximum number of level set functions we want to use before we start.

For many practical applications, such kind of a priori information is often available or is chosen according the measurements that are available to us. Also, to ensure ellipticity of equation (1), we need each q_i to be positive, that is, we assume that there exist $0 < a_i < b_i < \infty$ that are known a priori such that $q_i \in [a_i, b_i]$.

3. The parameter identification problem

We shall try to identify the coefficient q from a measurement of u on a subdomain $\hat{\Omega}$. We shall perform the numerical tests both for the case that $\hat{\Omega} = \Omega$ and the case that $\hat{\Omega} \subset \Omega$. In case that $\hat{\Omega} = \Omega$, existence and uniqueness of the inverse problem is already known. For general cases, studies about existence and uniqueness are still missing in the literature. In this work, the parameter identification problem is formulated as a least-square minimization problem and then we propose to use the augmented Lagrangian method to solve the least-squares minimization problem with the equation as constraint.

We start by defining the equation error of equation (1) as $e = e(q,u) \in H_0^1(\Omega)$ which is the variational solution of

$$(\nabla e, \nabla v) = (q\nabla u, \nabla v) - (f, v)\,, \quad \forall v \in H_0^1(\Omega). \tag{6}$$

Here and later $(\cdot,\cdot)$ denotes the L^2-innerproduct over Ω. We will also use the notation $\|\cdot\|$ to denote the asociated norm. For a given q and a given u, we say that they satisfy the equation (1) if and only if $e(q,u)=0$. In order to solve our inverse problem, we shall try to find a q and u such that $e(q,u)=0$ and also fits the measurements $\hat{u}$ best among all admissible functions q and u. Using the level set functions, the coefficient q will be represented as functions of the level set functions ϕ_j and the piecewise values q_i and the minimization problem we need to solve takes the form

$$\begin{array}{c}\min\limits_{q_i,\phi_j,u}\left(\frac{1}{2}\|u-\hat{u}\|^2_{L^2(\hat{\Omega})}+\beta\sum\limits_{i=1}^{n}\int_\Omega|\nabla H(\phi_j)|dx\right), \\ \text{under the conditions } e(q,u)=0, \quad |\nabla\phi_j|=1,\forall j.\end{array} \tag{7}$$

In the above q is a function of ϕ_j and q_i. The constraint $e(q,u)=0$ makes sure the equation error is zero. The first term tries to minimize the deviation between the calculated u and measured $\hat{u}$, while $\sum_{i=1}^{n}\int_\Omega|\nabla H(\phi_j)|dx$ in the second term is referred to as a regularization term. In case that Ω is a one-dimensional domain, then $\int_\Omega|\nabla H(\phi_j)|dx$ equals to the number of points that the level set functions ϕ_j equals zero. If Ω is two-dimensional, it is the length of the zero level set curves of ϕ_j. For three-dimensional cases, then it is the area of the zero level set surfaces of ϕ_j.

To solve (7), we use the augmented Lagrangian formulation, and the corresponding Lagrangian functional $L : R^{2^n}\times[Lip(\Omega)]^n\times H^1_0(\Omega)\times H^1_0(\Omega)\mapsto R$ is

$$\begin{array}{c}L(q_i,\phi_j,u,\lambda)=\frac{1}{2}\|u-\hat{u}\|^2_{L^2(\hat{\Omega})}+ \\ +\beta\sum\limits_{j=1}^{n}\int_\Omega|\nabla H(\phi_j)|dx+\frac{c}{2}\|\nabla e\|^2-(\nabla\lambda,\nabla e).\end{array} \tag{8}$$

The Lagrangian multiplier λ is only trying to enforce the equation constraint $e(q,u)=0$. The other constraints $|\nabla\phi_j|=1$ will be enforced by some other methods well developed for the level set methods. Due to the fact that the ϕ_j's are the viscosity solutions for the Eikonal equation, it is not easy to enforce them by the Lagrangian multiplier method.

In order to find a minimizer for the minimization problem (7), we shall use an algorithm of the type of the Lancelot method which will be given later in this section. The algorithm needs the derivatives of the Lagrangian functional with respect to the minimization variables.

3.1. Calculation of $\nabla_{q_i} L$

The derivative of L with respect to q_i is

$$\frac{\partial L}{\partial q_i} = c\left(\nabla e, \nabla \frac{\partial e}{\partial q_i}\right) - \left(\nabla \lambda, \nabla \frac{\partial e}{\partial q_i}\right) = \left(\nabla \frac{\partial e}{\partial q_i}, \nabla(ce - \lambda)\right).$$

From (6), it follows that the derivative of ∇e with respect to q_i is

$$\left(\nabla \frac{\partial e}{\partial q_i}, \nabla v\right) = \left(\frac{\partial q}{\partial q_i} \nabla u, \nabla v\right).$$

Taking v to be $ce - \lambda$ gives

$$\frac{\partial L}{\partial q_i} = \left(\frac{\partial q}{\partial q_i} \nabla u, \nabla(ce - \lambda)\right).$$

It is trivial to calculate the derivative of q with respect to q_i when using equation (5).

3.2. Calculation of $\nabla_{\phi_j} L$

For clarity of the presentation, we shall first calculate the Gateaux derivative of the regularization term, i.e. we first calculate the Gateaux derivative for the following functional

$$R(\phi_j) = \int_\Omega |\nabla H(\phi_j)| dx = \int_\Omega \delta(\phi_j) |\nabla \phi_j| dx.$$

Here and later, δ denotes the Dirac-function. To get the derivative of R with respect to ϕ_j in the direction μ_j, we proceed

$$\frac{\partial R}{\partial \phi_j} \cdot \mu_j = \int_\Omega \delta'(\phi_j) \mu_j |\nabla \phi_j| dx + \int_\Omega \delta(\phi_j) \frac{\nabla \phi_j}{|\nabla \phi_j|} \cdot \nabla \mu_j dx.$$

Applying Greens formula to the last term which can be theoretically verified by replacing the delta function by a smooth function and then passing to the limit, we will get that

$$\begin{aligned}
\frac{\partial R}{\partial \phi_j} \cdot \mu_j &= \int_\Omega \delta'(\phi_j) \mu_j |\nabla \phi_j| dx - \int_\Omega \nabla \cdot \left(\delta(\phi_j) \frac{\nabla \phi_j}{|\nabla \phi_j|}\right) \mu_j dx \\
&= \int_\Omega \delta'(\phi_j) \mu_j |\nabla \phi_j| dx - \int_\Omega \left(\delta'(\phi_j) \frac{|\nabla \phi_j|^2}{|\nabla \phi_j|} \mu_j + \delta(\phi_j) \mu_j \nabla \cdot \frac{\nabla \phi_j}{|\nabla \phi_j|}\right) dx \qquad (9) \\
&= -\int_\Omega \delta(\phi_j) \mu_j \nabla \cdot \frac{\nabla \phi_j}{|\nabla \phi_j|} dx,
\end{aligned}$$

which indicates that

$$\frac{\partial R}{\partial \phi_j} = -\delta(\phi_j)\nabla \cdot \frac{\nabla \phi_j}{|\nabla \phi_j|}.$$

Denote the Gateaux derivative of L with respect to ϕ_j in the direction μ_j as $\frac{\partial L}{\partial \phi_j} \cdot \mu_j$. The Gateaux derivative in this case is

$$\begin{aligned}\frac{\partial L}{\partial \phi_j} \cdot \mu_j &= c\left(\nabla e, \nabla\left(\frac{\partial e}{\partial \phi_j} \cdot \mu_j\right)\right) - \left(\nabla\lambda, \nabla\left(\frac{\partial e}{\partial \phi_j} \cdot \mu_j\right)\right) + \beta\frac{\partial R}{\partial \phi_j} \cdot \mu_j \\ &= \left(\nabla\frac{\partial e}{\partial \phi_j} \cdot \mu_j, c\nabla e - \nabla\lambda\right) + \beta\frac{\partial R}{\partial \phi_j} \cdot \mu_j.\end{aligned}$$

The derivative of e with respect to ϕ_j in the direction μ_j is

$$\left(\nabla\left(\frac{\partial e}{\partial \phi_j} \cdot \mu_j\right), \nabla v\right) = \left(\frac{\partial q}{\partial \phi_j} \cdot \mu_j \nabla u, \nabla v\right),$$

Taking v to be $ce - \lambda$, we get that

$$\frac{\partial L}{\partial \phi_j} \cdot \mu_j = \left(\frac{\partial q}{\partial \phi_j} \cdot \mu_j,\ \nabla u \cdot \nabla(ce - \lambda)\right) + \beta\frac{\partial R}{\partial \phi_j} \cdot \mu_j. \tag{10}$$

From (5), it is easy to calculate the Gateaux derivative $\frac{\partial q}{\partial \phi_j} \cdot \mu_j$. For simplicity of the presentation, let us take the case that we only have two level set functions. Then q takes the form (4). Consequently, the Gateaux derivative for the function ϕ_j in a direction μ_j is

$$\frac{\partial q}{\partial \phi_1} \cdot \mu_1 = \big[(q_1 - q_3)H(\phi_2) + (q_2 - q_4)(1 - H(\phi_2)\big]\delta(\phi_1)\mu_1 \tag{11}$$

$$\frac{\partial q}{\partial \phi_2} \cdot \mu_2 = \big[(q_1 - q_2)H(\phi_1) + (q_3 - q_4)(1 - H(\phi_1)\big]\delta(\phi_2)\mu_2. \tag{12}$$

3.3. Calculation of $\nabla_u L$

We perturb u to $u + \epsilon w$ and try to calculate the Gateaux derivative of L with u in the direction w. First note that

$$\left(\nabla\left(\frac{\partial e}{\partial u} \cdot w\right), \nabla v\right) = (q\nabla w, \nabla v), \quad \forall v \in H_0^1(\Omega),$$

and

$$\begin{aligned}\frac{\partial L}{\partial u} \cdot w &= (u - \hat{u}, w)_{L^2(\hat{\Omega})} + c\left(\nabla e, \nabla\left(\frac{\partial e}{\partial u} \cdot w\right)\right) - \left(\nabla\lambda, \nabla\left(\frac{\partial e}{\partial u} \cdot w\right)\right) \\ &= (u - \hat{u}, w)_{L^2(\hat{\Omega})} + \left(\nabla(ce - \lambda), \nabla\left(\frac{\partial e}{\partial u} \cdot w\right)\right).\end{aligned}$$

Combining the above two equalities, it is true that

$$\frac{\partial L}{\partial u} \cdot w = (u - \hat{u}, w)_{L^2(\hat{\Omega})} + (\nabla(ce - \lambda), q\nabla w) .$$

This indicates that

$$\frac{\partial L}{\partial u} = (u - \hat{u})\chi_{\hat{\Omega}} - \nabla \cdot (q\nabla(ce - \lambda)),$$

where $\chi_{\hat{\Omega}}$ is the characteristic function for the subdomain $\hat{\Omega}$, i.e. $\chi_{\hat{\Omega}}(x) = 1$ if $x \in \hat{\Omega}$ and $\chi_{\hat{\Omega}}(x) = 0$ if $x \notin \hat{\Omega}$.

3.4. An algorithm of Lancelot type

To solve the minimization problem (8) we will use a Lancelot type algorithm, as described in Conn, Gould and Toint [3]. In our case, the algorithm can be written as follows

Algorithm 1 *Choose q_i^0, ϕ_j^0, u^0 as initial guess for the solution, and set $\lambda^0 = 0$, $k = 0$. Also chose initial tolerances $\epsilon_m > 0$, $\epsilon_e > 0$, and the parameters $c > 0$, $\beta > 0$, $\omega > 1$.*

Then iteratively do the following steps:

1. *Find an approximative minimum $\left(q_i^{k+1}, \phi_j^{k+1}, u^{k+1}\right)$ of equation* (8) *such that $\left\|\nabla_{q_i^{k+1}, \phi_j^{k+1}, u^{k+1}} L\right\| \leq \epsilon_m$.*

2. *If $\|\nabla e\| < \epsilon_e$, update λ by $\lambda^{k+1} = \lambda^k - ce\left(q_i^{k+1}, \phi_j^{k+1}, u^{k+1}\right)$, else update c by $c \leftarrow \omega c$.*

3. *Decrease ϵ_m, ϵ_e, and set $k \leftarrow k + 1$.*

In the first step of the iteration, we actually try to solve the minimization problem. Since λ is not solved for, neither do we need to drive $\|\nabla L\|$ to zero. The second step checks to see if the equation error is sufficiently small, and if so is the case, λ may be updated. Should the error not be small, the penalty parameter c is increased, making the augmented Lagrangian functional more dominated by the $\|\nabla e\|^2$ term. With these steps, we can decrease the tolerances, and go to the next iteration.

For the minimization problem, one may chose any suitable method for nonlinear, unconstrained problems, such as the method of steepest descent, the non-linear conjugate gradients, or the Quasi-Newton methods. Steepest descent methods, while being simple to implement, are known for giving low performance, and the Quasi-Newton methods needs careful implementation and provisions to limit memory usage, but they often yield excellent performance. In our case, the nonlinear conjugate gradients method was found suitable. It has good performance, low memory usage, and is easy to implement. See [7] for more information regarding these methods.

There are some aspects of our minimization that require additional explanation. We compute the composite derivative of L, that is a vector consisting of $\nabla_{q_i^{k,l}}$, then $\nabla_{\phi_j^{k,l}}$ and finally $\nabla_{u^{k,l}}$. This makes us minimize L with all the unknowns simultaneously. The reason for doing this instead of optimizing each unknown separately is that the latter approach is equivalent to a coordinate descent method, giving poor convergence. However, there may be problems finding an optimal steplength for this composite search direction, ie. ϕ_j may require a smaller steplength than u. By defining

$$\tilde{u} = su, \ \ s > 0,$$

and using $\tilde{u}$ in the optimization, the rate of convergence may be improved. We have in our experiments chosen s experimentally.

We also need to enforce constraints on q_i and ϕ_j. The former is just to set $q_i = \min\{\max\{q_i, a_i\}, b_i\}$, while for the latter we would try to ensure that

$$|\nabla \phi_j^{k+1}| = 1, \quad \phi_j^{k+1} = 0 \text{ on } \Gamma_j^k.$$

For a one-dimensional problem, this can easily be done, but in higher dimensions we can use methods described in Osher and Fedkiw [9], and Smereka, Sussman, Fatemi and Osher [6]. There are fast and cheap algorithms to solve this problem, see [6, 9].

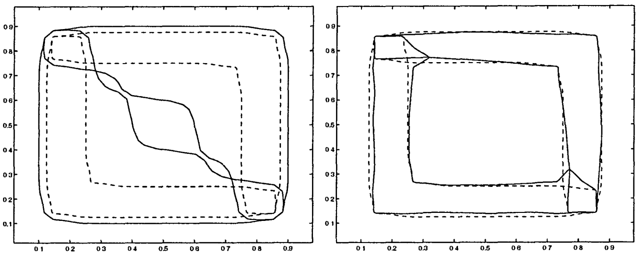
4. Numerical experiments

In our numerical tests, we will consider three cases. First, the q_i's are all known, and we try to identify ϕ_j and u; then we perturb q_i, and attempt to identify ϕ_j and u with q_i fixed. Thirdly, we add noise to $\hat{u}$, and try to identify ϕ_j and u while q_i is known.

The equation we will use is

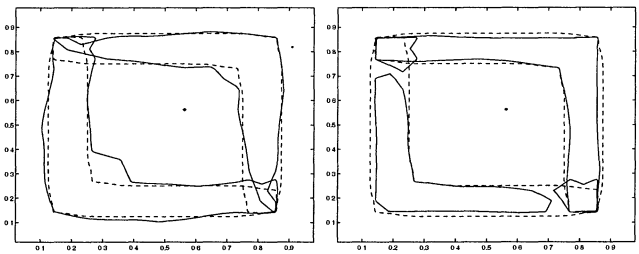
$$\begin{aligned} -\nabla \cdot (q \nabla u) &= 2\pi^2 \sin(\pi x) \sin(\pi y), && \text{in } \Omega \\ u &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{13}$$

Here, $\Omega = (0,1) \times (0,1)$. All experiments are done with a uniform, 2D mesh of Ω, with 24×24 elements. The numerical parameters used are $\beta = 10^{-7}$, $c = 5 \cdot 10^{-6}$, $\omega = 1.1$, $s = 25$. All the Figures show the zero level sets of the exact and of the computed ϕ_1 (in the lower-left corner) and ϕ_2 (in the upper-right corner) in the various situations that we have considered.



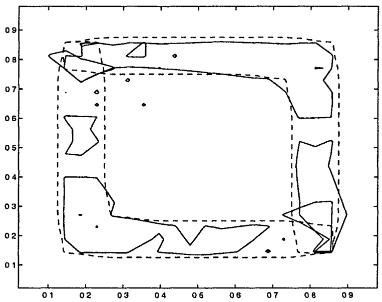
(a) Initial (b) Final

Figure 1. Drawing of the zero level set curves for equation (13). The dashed lines are the exact zero level sets, and the solid lines are the computed solution. This is the first case, and convergence was attained after about 300 iterations. Also, $\|u_h - u\|_2 \approx 1.1906 \cdot 10^{-4}$ at time of convergence.

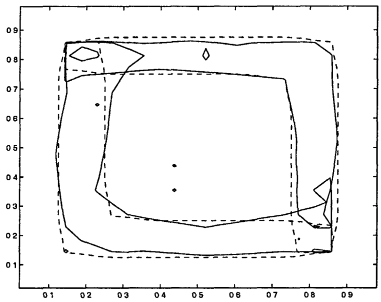


(a) Coefficient q^- (b) Coefficient q^+

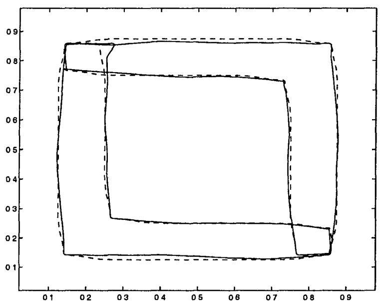
Figure 2. The dashed lines are the exact zero level sets, and the solid lines are the computed solution. This is the second case, with perturbed q_i's and we start with the same initial levelsets as in the first case. For q^-, it took about 350 iterations to converge, and $\|u_h - u\|_2 \approx 1.3027 \cdot 10^{-4}$ at that time, while for q^+ it only took about 100 iterations, and here $\|u_h - u\|_2 \approx 1.1182 \cdot 10^{-4}$. Since the discontinuities are smaller in this latter case, this quicker convergence is expected.



(a) Noise amplitude 10^{-2}



(b) Noise amplitude $5 \cdot 10^{-3}$



(c) Noise amplitude 10^{-3}

Figure 3. The dashed lines are the exact zero level sets, and the solid lines are the computed solution. Here is the third case, with noise added to $\hat{u}$. After about 100 iterations, we had convergence, and the error $\|u_h - u\|_2$ was about $1.7251 \cdot 10^{-4}$, $1.5019 \cdot 10^{-4}$ and $1.4223 \cdot 10^{-4}$ for noise amplitudes of $10^{-2}, 5 \cdot 10^{-3}$ and 10^{-3} respectively. More noise made it generally impossible to get convergence, and with less noise the solution was not distinguishable from the case without noise.

We shall use q given as follows. $q_1 = 1$ when $\phi_1, \phi_2 \leq 0$, $q_2 = 1 + 2^{1/3}$ when $\phi_1 \leq 0, \phi_2 > 0$, $q_3 = 1 + 2^{2/3}$ when $\phi_1 > 0, \phi_2 \leq 0$, and finally $q_4 = 3$ when $\phi_1, \phi_2 > 0$. ϕ_1 is positive within the union of the following rectangles

$$\{x, y{:}3/4{<}x{<}7/8, 1/8{<}y{<}7/8\} \cup \{x, y : 1/8{<}x{<}7/8, 3/4{<}y{<}7/8\},$$

and ϕ_2 is likewise positive within this union

$$\{x, y{:}1/8{<}x{<}1/4, 1/8{<}y{<}7/8\} \cup \{x, y : 1/8{<}x{<}7/8, 1/8{<}y{<}1/4\}.$$

In all cases, we let $\hat{\Omega}$ extend from the boundary of Ω and into the interior by a length of 1/3 from all sides. In the remainder, we let $\hat{\Omega}$ be coarser than Ω by a factor of two. Thus we have complete observations $\hat{u}$ along the boundary, and coarser observations in the interior. Note that $\|\nabla u\| \approx 0$ near the center of Ω. Because of this, we cannot easily identify q in the center since for $\|\nabla u\| = 0$, q is no longer unique.

Solving for the first case yields the results in Figure 1, which are quite accurate.

For the second case, we will use the two sets of q_i coefficients q^- and q^+, and they are given as follows. $q_1^\pm = q_1 \pm 0.1$, $q_2^\pm = q_2 \pm 0.051$, $q_3^\pm = q_3 \mp 0.05$, $q_4^\mp = q_4 \pm 0.1$.

Note that the perturbations in q^- create larger jumps, while in q^+ the jumps are smaller. The results are in Figure 2, and we see that it is easier to identify q^+ rather than q^- due to smaller jumps. Also note that the deviations from Figure 1 are small.

We now come to the third case where we will add noise to our observations, and try find ϕ_j and u. Adding normally distributed noise of varying magnitude gives us the results in Figure 3. Having larger noise-magnitude than 10^{-2} makes it generally hard to get convergence, while a noise-magnitude of less than 10^{-3} hardly makes much of an impact on our case.

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