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Fluctuations for a $\nabla\phi$ interface model with repulsion from a wall

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Abstract. We consider a $\nabla\phi$ interface model on a one-dimensional lattice with repulsion from a hard wall. We suppose that the repulsion is of the form $c\phi^{-\alpha-1}$, where $c, \alpha > 0$ and ϕ denotes the height of the interface from the wall. We prove convergence of the equilibrium fluctuations around the hydrodynamic limit to the solution of a SPDE with singular drift. If $c \to 0$ the system becomes the Funaki-Olla $\nabla\phi$ interface model with reflection at the wall, whose equilibrium fluctuations converge to the solution of a SPDE with reflection. We give a new proof of this result using the characterization of such solution as the diffusion generated by an infinite dimensional Dirichlet Form, obtained in a previous paper. Our method is based on a study of integration by parts formulae w.r.t. the equilibrium measure of the interface model and allows to avoid the proof of the so called Boltzmann-Gibbs principle. We also obtain convergence of finite dimensional distributions of non-equilibrium fluctuations around the stationary hydrodynamic limit 0.

1. Introduction

This paper concerns fluctuations of an interface near a hard wall. The system is defined on the one-dimensional lattice $\Gamma_N := \{1, 2, \dots, N\}$ and the location of the interface at time t is represented by the height variables $\phi_t = \{\phi_t(x), x \in \Gamma_N\} \in \Omega_N^+ := [0, \infty)^{\Gamma_N}$ measured from the wall Γ_N.

We consider two distinct behaviors of the microscopic interface near the wall: reflection or repulsion. In the first case the evolution is determined by the SDE of Skorohod's type with normal reflection at the boundary of Ω_N^+:

$$d\phi_t(x) = -\frac{1}{2}\{V'(\phi_t(x) - \phi_t(x-1)) - V'(\phi_t(x+1) - \phi_t(x))\}\,dt + dl_t(x) + dw_t(x), \qquad x \in \Gamma_N, \tag{1}$$

in the second case, for $c, \alpha > 0$, by the SDE:

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$$d\phi_t(x) = -\frac{1}{2}\{V'(\phi_t(x) - \phi_t(x-1)) - V'(\phi_t(x+1) - \phi_t(x))\} dt$$
$$+ N^{\frac{\alpha-2}{2}} \frac{c\alpha}{2} \frac{1}{[\phi_t(x)]^{\alpha+1}} dt + dw_t(x), \qquad x \in \Gamma_N, \tag{2}$$

where in both equations $(w(x) : x \in \Gamma_N)$ is an independent family of Brownian motions, we set $\phi_t(0) = \phi_t(N+1) = 0$ as boundary conditions at $\partial\Gamma_N := \{0, N+1\}$ and the potential $V \in C^2(\mathbb{R})$ is assumed to be even and strictly convex, i.e. $c_- \le V'' \le c_+$ for some $c_-, c_+ > 0$.

In (1) the reflecting process $t \mapsto l_t(x)$ prevents $\phi_t(x)$ from becoming negative. In (2) the drift $[\phi(x)]^{-1-\alpha}$ tends to $+\infty$ as $\phi(x) \to 0$ and every solution of (2) satisfies $\phi_t(x) > 0$ for all $t > 0$ and $x \in \Gamma_N$. We call the solution of (1) and (2) a $\nabla\phi$ interface model with reflection, respectively repulsion from the wall. As $\alpha \downarrow 0$ the solution of (2) tends to the solution of (1) with the same initial condition: with an abuse of notation we interpret equation (1) as a particular case of equation (2) with $\alpha = 0$.

The model (1) has been already considered by T. Funaki and S. Olla in [8]. In this case the natural stationary distribution of the interface is the probability measure on $[0, \infty)^{\Gamma_N}$:

$$\mu_N(d\phi) := \frac{1}{Z_N} \exp(-H_N(\phi)) \prod_{x \in \Gamma_N} 1_{(\phi(x) \ge 0)} \, d\phi(x),$$

where Z_N is a normalization constant and H_N is the Hamiltonian:

$$H_N(\phi) := \sum_{x=1}^{N+1} V(\phi(x) - \phi(x-1)), \quad \phi(0) := \phi(N+1) := 0.$$

Notice that μ_N is a Gibbs measure conditioned on $[0, \infty)^{\Gamma_N}$: see e.g. [3] for related results. The stochastic dynamics defined by (2) admits as stationary distribution the probability measure on $(0, \infty)^{\Gamma_N}$:

$$\mu_N^{c,\alpha} := \frac{1}{Z_N^{c,\alpha}} \exp\left(-N^{\frac{\alpha-2}{2}} \sum_{x \in \Gamma_N} \frac{c}{[\phi(x)]^\alpha}\right) \mu_N(d\phi).$$

Notice that the density at ϕ of $\mu_N^{c,\alpha}$ w.r.t. μ_N tends to 0 if $\phi(x) \to 0$ for some $x \in \Gamma_N$. Moreover, $\mu_N^{c,\alpha} \to \mu_N$ as $\alpha \to 0$.

Suppose now that $(\phi_t)_{t \ge 0}$ is the unique stationary solution of (1), respectively (2). We introduce the macroscopic height variable:

$$h_N(t, \theta) := \frac{1}{N} \phi_{N^2 t}(\lfloor N\theta \rfloor + 1), \qquad \theta \in [0, 1), \ t \ge 0,$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Then h_N converges to 0 in probability as $N \to \infty$ and it is interesting to study the fluctuation field Φ_N:

$$\Phi_N(t, \theta) := \sqrt{N}\, h_N(t, \theta) = \frac{1}{\sqrt{N}} \phi_{N^2 t}(\lfloor N\theta \rfloor + 1), \quad \theta \in [0, 1). \tag{3}$$

The aim of this paper is to prove convergence results for the law of the equilibrium fluctuation field Φ_N. For all $c > 0$, as $N \to \infty$, Φ_N converges in law in a suitable space of distributions to the unique stationary in time and non-negative solution $(u(t,\theta), t \geq 0, \theta \in [0,1])$ of the Stochastic PDE:

$$\alpha \geq 2: \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{q}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{c\alpha}{2}\dfrac{1}{u^{1+\alpha}} + \dfrac{\partial^2 W}{\partial t \partial \theta} \\ u(t,0) = u(t,1) = 0, \quad t \geq 0 \end{cases} \tag{4}$$

$$\alpha \in [0,2): \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{q}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{c\alpha}{2}\dfrac{1}{u^{1+\alpha}} + \dfrac{\partial^2 W}{\partial t \partial \theta} + \zeta \\ u(t,0) = u(t,1) = 0, \quad t \geq 0 \\ u \geq 0, \ d\zeta \geq 0, \ \int u \, d\zeta = 0 \end{cases} \tag{5}$$

where W is a Brownian Sheet in $[0,\infty) \times [0,1]$ and:

$$q^{-1} := \frac{1}{\kappa}\int_{\mathbb{R}} r^2 e^{-V(r)}\, dr, \qquad \kappa := \int_{\mathbb{R}} e^{-V(r)}\, dr.$$

Notice that (5) is a SPDE with reflection of the Nualart-Pardoux type (see [11]), i.e. ζ is a measure on $(0,\infty) \times (0,1)$, forcing u to remain non-negative. Moreover, in (4) and (5) we have, for $\alpha > 0$, $u^{-1-\alpha} \in L^1_{loc}([0,\infty) \times (0,1))$, which makes the unbounded non-linearity meaningful even at the points where u vanishes. Existence and uniqueness of solutions of (4) and (5) have been proved in [17].

Notice that for $\alpha \in (0,2)$ the discrete variable $\phi_t(x)$ is a.s. strictly positive for all $t > 0$, while the continuum variable $u(t,\theta)$ a.s. hits the wall for some $t \geq 0$ and need afterwards the reflection term ζ in order to remain non-negative. For $\alpha \geq 2$ the repulsive drift $u^{-1-\alpha}$ is strong enough to make the reflection term unnecessary. In the critical case $\alpha = 2$, the rescaling $N^{\frac{\alpha-2}{2}}$ in (2) becomes constant; the drift $[\phi_t(x)]^{-3}$ is invariant under the rescaling (3) which defines Φ_N. In this case the stationary distribution for (4) can be explicitly computed and is equal to the law of $q^{-1/2}$ times a Bessel bridge of dimension δ between 0 and 0 over $[0,1]$, where $\delta > 3$ is defined by $(\delta-1)(\delta-3) = 8cq$, see [17].

We recall that the convergence of the fluctuations field for (1) has already been proved by Funaki and Olla in [8]. The main technique in [8] is the penalization: the reflection at the wall is substituted by a strong drift of the form $(\phi_t(x))^-/\epsilon$, $\epsilon > 0$, the fluctuation result is proved for such equation, and finally the monotonicity in ϵ allows to conclude the result for the interface on the wall. The main difficulty is the proof of the so called Boltzmann-Gibbs Principle.

This paper proposes a new proof of Funaki-Olla's result and more generally a new approach to convergence of fluctuations of reflected interfaces. First we characterize the limit equation in terms of a simpler object, namely a Dirichlet Form (a *static* object); then, in order to identify the weak limits of $(\Phi_N)_N$, we prove that the limits of the associated resolvent operators are the resolvent of the limit.

We use two main technical tools. The first one is a static information, i.e. concerning only the unique invariant measure μ_N, defined above, of the reflected interface. We call this step a *static Boltzmann-Gibbs Principle*: we prove convergence of the non-linear discrete Laplacian to the linear continuum Laplacian in a weak sense w.r.t. μ_N: see Theorem 1 below. This is obtained by proving that an integration by parts formula (IbPF) for μ_N converges, under rescaling, to the IbPF for the Bessel bridge of dimension 3, first proved in [16]. In particular, the finite-dimensional boundary term converges to the infinite-dimensional one, giving a new independent proof of the IbPF for the 3-d Bessel bridge, based on the invariance principle.

The second tool is a dynamical information, consisting in an estimate, independent of N, on a smoothing property of the transition semigroup of the reflected interface: see Lemma 5 below. This formula is based on a random walk representation for the gradient of the transition semigroup, introduced in [4] and extended to reflected SDEs in [5], and on an upper bound for the kernel of time-inhomogeneous random walks, proved in [2].

This approach presents two main advantages: we bypass the difficult proof of the Boltzmann-Gibbs principle and we make no use of monotonicity properties. Moreover we prove directly the convergence for the fluctuation field of the reflected interface, rather than for approximating processes. This seems to be promising for applications to interfaces with different behavior at the wall, like sticky reflection, for which monotonicity possibly fails.

The convergence of the fluctuations of (2) follows from the result for the reflected case, adding to equation (1) a drift of the form $c\alpha/[2(\epsilon+\phi(x))^{1+\alpha}]$, $\epsilon > 0$, and passing to the limit as $\epsilon \to 0$, using mainly the techniques of [8].

Our method gives also convergence of finite-dimensional distributions of non-stationary fluctuations around the stationary hydrodynamic limit 0: we consider solutions of (1), resp. (2), with arbitrary initial distribution converging weakly in $L^2(0, 1)$ under the scaling (3). The macroscopic height h_N still tends to 0, but the fluctuation field Φ_N is no more stationary. The limit equation is the same as for the equilibrium case: see point 2 in Theorems 2 and 3.

The paper is organized as follows. After introducing the reflected interface in section 2, we prove the so-called static Boltzmann-Gibbs principle in section 3 and we give a smoothing property of the transition semigroup of the reflected interface in section 4. In section 5 and 6 we prove the convergence of the fluctuations of the interface with reflection and, respectively, repulsion from the wall.

We fix some notations. We introduce the Hilbert space $H := L^2(0, 1)$ with the canonical scalar product $\langle \cdot\,, \cdot \rangle$ and associated norm $\| \cdot \|$. We define:

$$\Gamma_N := \{1, \dots, N\}, \qquad \Omega_N^+ := [0, \infty)^{\Gamma_N}, \qquad K := \{k \in H : k \geq 0\}.$$

We denote by C_b, respectively C_b^1, spaces of bounded uniformly continuous functions, resp. bounded and uniformly continuous together with the first Fréchet derivative. We denote by $C_c(0, 1)$ the set of continuous functions with compact support in $(0, 1)$, by $C_c^2(0, 1)$ the set of twice continuously differentiable functions in $C_c(0, 1)$ and by $D([0, 1])$ the set of càdlàg functions from $[0, 1]$ to $\mathbb{R}$, endowed with the Skorohod topology. We define $\mathrm{Exp}(H)$ as the linear span of $\{\cos(\langle h, \cdot\rangle), \sin(\langle h, \cdot\rangle) :$

$h \in C_c^2(0,1)\}$. If $J \subseteq H$ is a closed linear subspace, $f \in C_b^1(J)$ and $h \in J$ we set

$$\partial_h f(x) := \langle \nabla f(x), h \rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} f(x+\epsilon h), \quad x \in J.$$

2. The reflected interface

Let $\{(w_t(x))_{t\geq 0} : x \in \mathbb{N}\}$ be an independent sequence of standard Brownian motions and $U : [0,\infty) \mapsto [0,\infty)$ be bounded, convex, continuously differentiable and monotone non-increasing. The dynamics of $(\phi_t(x) : x \in \Gamma_N)_{t\geq 0}$, height from the wall of the reflected interface, is governed by the stochastic differential equation of the Skorohod type:

$$\phi_t(x) = \phi_0(x) - \frac{1}{2}\int_0^t \left\{V'(\phi_s(x) - \phi_s(x-1)) - V'(\phi_s(x+1) - \phi_s(x))\right\} ds$$
$$+ l_t(x) - \frac{N^{-3/2}}{2}\int_0^t U'\left(N^{-1/2}\phi_s(x)\right) ds + w_t(x), \tag{6}$$

for all $x \in \Gamma_N$, subject to the conditions:

$$\phi_t(x) \geq 0, \qquad t \mapsto l_t(x) \text{ continuous and non}-\text{decreasing}, \qquad l_0(x) = 0,$$

$$\int_0^\infty \phi_t(x)\, dl_t(x) = 0, \qquad x \in \Gamma_N,$$

and to the boundary conditions at $\partial\Gamma_N := \{0, N+1\}$:

$$\phi_t(0) = \phi_t(N+1) = 0, \qquad t \geq 0.$$

The Funaki-Olla model (1) corresponds to $U \equiv 0$. In section 6 approximations of the solution of (2) are constructed by means of smooth potentials U. Throughout the paper the potential V satisfies the following conditions:

(V1) $V \in C^2(\mathbb{R})$,
(V2) (symmetry), $V(-r) = V(r)$, $r \in \mathbb{R}$,
(V3) (strict convexity), $c_- \leq V''(r) \leq c_+$, $r \in \mathbb{R}$, for some $c_-, c_+ \in (0,\infty)$.

By (V3), $\exp(-V)$ is integrable over $\mathbb{R}$. For notational convenience we also suppose that:

(V4) (normalization), $\int_{\mathbb{R}} \exp(-V(r))\, dr = 1$.

Notice that the normalization (V4) does not affect equations (2) and (6), where only V' appears.

Lemma 1. *For all $\phi_0 \in K_N$ there exists a unique pair $(\phi_t, l_t)_{t\geq 0}$, solution of* (6). *We set $\phi^U(t,\phi_0) := \phi_t$, $t \geq 0$. For all $t \geq 0$ and $\phi_0, \overline{\phi}_0 \in \Omega_N^+$:*

$$\sum_{x\in\Gamma_N} \left|\phi^U(t,\phi_0)(x) - \phi^U(t,\overline{\phi}_0)(x)\right|^2 \leq e^{-\frac{c_- t}{N^2}} \sum_{x\in\Gamma_N} \left|\phi_0(x) - \overline{\phi}_0(x)\right|^2. \tag{7}$$

Proof. The existence follows from [14], the uniqueness from (7), which we prove now. We set for $f \in \mathbb{R}^{\Gamma_N}$, $D_N f(x) := f(x+1) - f(x)$, $x = 0, \dots, N$, with $f(0) := f(N+1) := 0$. Let (ϕ, l) and $(\overline{\phi}, \overline{l})$ be solutions of (6) with initial condition ϕ_0, resp. $\overline{\phi}_0$. Setting $\psi_t := \phi_t - \overline{\phi}_t$, by Itô's formula we obtain:

$$\begin{aligned}\sum_{x\in\Gamma_N} d[\psi_t(x)]^2 &= -\sum_{x=0}^{N} D_N\psi_t(x)\left[V'(D_N\phi_t(x)) - V'(D_N\overline{\phi}_t(x))\right] dt \\ &\quad - \sum_{x\in\Gamma_N} \psi(x) N^{-3/2}\left(U'(N^{-1/2}\phi_s(x)) - U'(N^{-1/2}\overline{\phi}_s(x))\right) dt \\ &\quad + \sum_{x\in\Gamma_N} 2\psi(x)(dl_t(x) - d\bar{l}_t(x)) \\ &\leq -c_- \sum_{x=0}^{N} [D_N\psi_t(x)]^2 dt \leq -\frac{c_-}{N^2}\sum_{x\in\Gamma_N}[\psi_t(x)]^2\, dt,\end{aligned}$$

since U' is monotone non-decreasing and for any real $a_1, \dots a_N$ and $a_0 = 0$:

$$\sum_{i=1}^{N}(a_i)^2 = \sum_{i=1}^{N}\left[\sum_{j=1}^{i}(a_j - a_{j-1})\right]^2 \leq N^2 \sum_{i=1}^{N}(a_i - a_{i-1})^2. \qquad \square$$

We define the following probability measures for $a \geq 0$:

$$d\mu_{N,a} = \frac{1}{Z_{N,a}} \exp\left\{-H_{N,a}(\phi)\right\} \prod_{x\in\Gamma_N} 1_{(\phi(x)\geq 0)}\, d\phi(x)$$

where for $\phi \in \Omega_N^+$, $H_{N,a}$ is the Hamiltonian :

$$H_{N,a}(\phi) := \sum_{x\in\Gamma_N} V(\phi(x) - \phi(x-1)), \qquad \phi(0) := \phi(N+1) := \sqrt{N}\, a$$

and $Z_{N,a}$ is a normalization constant. Moreover we set:

$$\mu_N := \mu_{N,0}, \qquad Z_N := Z_{N,0}. \tag{8}$$

We consider a sequence of i.i.d. real random variables $(X_i)_{i\in\mathbb{N}}$, such that X_i has density $\exp(-V)dr$ on $\mathbb{R}$. We set:

$$q := \left(\mathbb{E}\left[X_1^2\right]\right)^{-1}. \tag{9}$$

For $n \in \mathbb{N}$ we set $S_n := X_1 + \cdots + X_n$, $X_0 := 0$. We denote by P_N the law of $(S_1, \ldots, S_N)$ under the conditioning $\{S_{N+1} = 0\}$. Then $\mu_N = P_N(\cdot \mid \Omega_N^+)$. We also set:

$$d\mu_N^U := \frac{1}{Z_N^U} \exp\left\{ -\frac{1}{N} \sum_{x \in \Gamma_N} U\left(N^{-1/2}\,\phi(x)\right) \right\} d\mu_N(\phi).$$

The Markov process $(\phi^U(t, \phi_0))_{t \geq 0, \phi_0 \in \Omega_N^+}$ is the diffusion generated by the symmetric Dirichlet Form in $L^2(\Omega_N^+, \mu_N^U)$, closure of:

$$C_b^1(\Omega_N^+) \ni F \mapsto e^{N,U}(F, F) := \frac{1}{2} \int_{\mathbb{R}_+^{\Gamma_N}} \sum_{x \in \Gamma_N} \left| \frac{\partial F}{\partial \phi(x)} \right|^2 d\mu_N^U,$$

and ϕ^U is reversible w.r.t. its unique invariant probability measure μ_N^U, see [7]. We denote by $(\phi^U(t))_{t \geq 0}$ the unique stationary solution of (38).

For all $N \in \mathbb{N}$ we define the map $\Lambda_N : \mathbb{R}^{\Gamma_N} \mapsto H$:

$$\Lambda_N(\phi)(\theta) := \frac{1}{\sqrt{N}}\, \phi(\lfloor N\theta \rfloor + 1), \quad \theta \in [0, 1),$$

where $\lfloor \cdot \rfloor$ denotes the integer part, and we define the spaces

$$H_N := \Lambda_N(\mathbb{R}^{\Gamma_N}) \subset H, \qquad K_N := \Lambda_N(\Omega_N^+) = K \cap H_N.$$

We denote by $1_{I(x)}$ the indicator function of $I(x)$, where

$$I(0) = I(N+1) := \emptyset, \qquad I(x) := [(x-1)/N, x/N), \quad x \in \Gamma_N.$$

Notice that H_N can be identified with the space of functions on $[0, 1)$ being constant on $I(x)$ for all $x \in \Gamma_N$ and K_N with the set of non-negative elements of H_N. Finally, we denote by $\Pi_N : H \mapsto H_N$ the orthogonal projection.

For all $k \in K_N$ and $t \geq 0$ we define now the rescaled reflected interface Φ_N^U and the associated invariant measure m_N^U:

$$\Phi_N^U(t, k) := \Lambda_N\left(\phi^U\left(N^2 t, \Lambda_N^{-1}(k)\right)\right), \qquad \Phi_N^U(t) := \Lambda_N\left(\phi^U\left(N^2 t\right)\right),$$

$$m_N := \Lambda_N^*(\mu_N), \qquad m_N^U := \Lambda_N^*(\mu_N^U) = \frac{1}{Z_N^U}\, e^{-\langle U(k), 1\rangle}\, m_N(dk).$$

Notice that $N \cdot \Lambda_N$ is a linear isometry between Hilbert spaces, i.e.

$$\|\Lambda_N(\phi)\|^2 = \int_0^1 |\Lambda_N(\phi)(\theta)|^2\, d\theta = \frac{1}{N^2} \sum_{x \in \Gamma_N} |\phi(x)|^2, \quad \phi \in \mathbb{R}^{\Gamma_N}. \tag{10}$$

For all $f \in C_b^1(H_N)$ we have $f \circ \Lambda_N \in C_b^1(\mathbb{R}^{\Gamma_N})$ and:

$$\sum_{x \in \Gamma_N} \left| \frac{\partial (f \circ \Lambda_N)}{\partial \phi(x)} \right|^2 = \frac{1}{N^2} \, \|(\nabla f) \circ \Lambda_N\|^2. \tag{11}$$

By (11), Φ_N^U is the diffusion generated by the symmetric Dirichlet Form $\mathcal{E}^{N,U}$ in $L^2(K_N, m_N^U)$:

$$\mathcal{E}^{N,U}(f, f) := \frac{1}{2} \int_{K_N} \|\nabla f\|^2 \, dm_N^U = N^2 \, e^{N,U}(f \circ \Lambda_N, f \circ \Lambda_N).$$

3. Static Boltzmann-Gibbs Principle

Let now $(e_\tau)_{\tau \in [0,1]}$ be the normalized Brownian excursion, i.e. the Bessel bridge of dimension 3 between 0 and 0 over [0, 1]: see [13]. We denote by m the law of $q^{-1/2}e$, where q is defined by (9). The relevance of m in our contest is made clear by the following result, proven at the end of the section.

Lemma 2. *m_N converges weakly in the Skorohod topology to m.*

This section is devoted to the proof of the following:

Theorem 1. *For all $h \in H$ there exist a Lipschitz map $\beta_h^N : K_N \mapsto \mathbb{R}$ and a finite measure Σ_h^N concentrated on ∂K_N, topological boundary of K_N, such that for all $f \in C_b^1(H)$*

$$\int_{K_N} \partial_{(\Pi_N h)} f \, dm_N = -\int_{K_N} \beta_h^N \, f \, dm_N - \int_{K_N} f \, d\Sigma_h^N. \tag{12}$$

Since $m_N(\partial K_N) = 0$, (β_h^N, Σ_h^N) is unique. Moreover for all $f \in C_b(H)$:

$$\lim_{N \to \infty} \int_{K_N} \beta_h^N \, f \, dm_N = \int_K \beta_h \, f \, dm, \qquad \forall \, h \in C_c^2(0,1), \tag{13}$$

$$\lim_{N \to \infty} \int_{K_N} f \, d\Sigma_h^N = \int_K f \, d\Sigma_h, \qquad \forall \, h \in C_c(0,1) \tag{14}$$

where $\beta_h : K \mapsto \mathbb{R}$,

$$\beta_h(k) := q\langle k, h'' \rangle, \quad \Sigma_h(dk) := \int_0^1 \frac{q^{1/2} \, h(r)}{\sqrt{2\pi r^3 (1-r)^3}} \, m(dk \mid k(r) = 0) \, dr.$$

Theorem 1 is applied in Section 5 to the proof of the convergence of the fluctuation field Φ_N^U. Formula (13) is the *static Boltzmann-Gibbs principle* mentioned in the introduction: β_h^N is the scalar product between h and the non-linear drift of the discrete SPDE satisfied by the rescaled interface Φ_N^U, β_h is clearly the scalar product between h and the linear drift of (5) with $\alpha = 0$, and (13) states that β_h^N converges to β_h in a weak sense.

Notice that (12) is an Integration by Parts Formula (IbPF) for m_N. We recall that in [16] the following IbPF for m is proved:

$$\int_K \partial_h f \, dm \;=\; -\int_K \beta_h \, f \, dm - \int_K f \, d\Sigma_h, \tag{15}$$

for all $h \in C_c^2(0,1)$, $f \in C_b^1(H)$. Therefore Theorem 1 is a strengthening of the invariance principle of Lemma 2: the law of the random walk with jumps distribution $\exp(-V)$, V convex, conditioned to be non-negative, induces an IbPF, which converges in the scaling limit to the IbPF of m.

Theorem 1 also gives a new proof of (15). Indeed, in the particular case $V(r) = V_0(r) = r^2/2$, we have for $k \in K$, $h \in C_c^2(0,1)$:

$$\beta_h^N(k) \;=\; N^3 \sum_{x\in\Gamma_N} \langle k, 1_{I(x)}\rangle \, \langle h, 1_{I(x+1)} + 1_{I(x-1)} - 2\,1_{I(x)}\rangle \;\stackrel{N\to\infty}{\longrightarrow}\; \langle k, h''\rangle$$

and (13) follows easily from the convergence of m_N to m and from the estimate given in Lemma 4 below. Moreover the proof of (14) is direct for any choice of V. Therefore, (15) follows from (13)-(14) for $V = V_0$ and from Lemma 2.

However, in the general case $V \neq V_0$ our proof of (13) is not direct, but follows from (14), (15) and the convergence of the l.h.s. of (12).

Let us prove Theorem 1. First we compute explicitly the IbPF for μ_N and m_N. Then we prove (14) by explicit computations on Σ_h^N and Σ_h and finally we deduce (13) with a density argument.

Let us start with the computation of the IbPF for μ_N. Notice that μ_N is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+^N$, with a smooth strictly positive density $\exp(-H_{N,0})/Z_N$. The classical Divergence Theorem gives for all $F \in C_b^1(\Omega_N^+)$ and $a \in \mathbb{R}^{\Gamma_N}$:

$$\int_{\Omega_N^+} \sum_{x\in\Gamma_N} a(x)\, \frac{\partial F}{\partial \phi(x)} \, d\mu_N \tag{16}$$

$$= \int_{\Omega_N^+} \sum_{x=1}^{N+1} (a(x) - a(x-1)) \, V'(\phi(x) - \phi(x-1)) \, F(\phi) \, \mu_N(d\phi)$$

$$- \sum_{x\in\Gamma_N} \frac{a(x)}{Z_N} \int_{[0,\infty)^{\Gamma_N\setminus\{x\}}} F(\phi^x) \exp\left\{-H_{N,0}(\phi^x)\right\} \prod_{y\in\Gamma_N\setminus\{x\}} 1_{(\phi(y)\geq 0)} d\phi(y),$$

where $a(0) := a(N+1) := 0$, $\phi^x(y) := \phi(y)$ if $y \neq x$ and $\phi^x(x) := 0$. Notice that the last term in the r.h.s. of (16) is a boundary term and by the definition of $H_{N,0}$ it can be written as:

$$-\sum_{x\in\Gamma_N} a(x)\frac{Z_{x-1}Z_{N-x}}{Z_N}\int_{\Omega_{x-1}\times\Omega_{N-x}} F\left((\phi_1,0,\phi_2)\right)\mu_{x-1}(d\phi_1)\otimes\mu_{N-x}(d\phi_2), \tag{17}$$

where for $(\phi_1,\phi_2)\in\Omega^+_{x-1}\times\Omega^+_{N-x}$, $(\phi_1,0,\phi_2)\in\Omega^+_N$ and for $y\in\Gamma_N$:

$$(\phi_1,0,\phi_2)(y) := \phi_1(y)\,1_{(y\le x-1)} + 0\cdot 1_{(y=x)} + \phi_2(y-x)\,1_{(y\ge x+1)}.$$

By the definition of Λ_N, for all $f\in C^1_b(H)$ and $h\in H$:

$$\left[\partial_{(\Pi_N h)} f\right](\Lambda_N(\phi)) = N^{3/2}\sum_{x\in\Gamma_N}\langle h, 1_{I(x)}\rangle\,\frac{\partial(f\circ\Lambda_N)}{\partial\phi(x)}(\phi). \tag{18}$$

Then by (16), (17) and (18) we have that the IbPF for m_N (12) holds with

$$\beta^N_h(k) := -N^{3/2}\sum_{x=1}^{N+1}\langle h, 1_{I(x)}-1_{I(x-1)}\rangle\, V'\left(N^{3/2}\langle k, 1_{I(x)}-1_{I(x-1)}\rangle\right),$$

$$\Sigma^N_h := N^{3/2}\sum_{x\in\Gamma_N}\langle h, 1_{I(x)}\rangle\,\frac{Z_{x-1}\,Z_{N-x}}{Z_N}\,T^*_{N,x}\left[m_{x-1}\otimes m_{N-x}\right], \tag{19}$$

where $T_{N,x}: K\times K\mapsto K$ and for $\tau\in[0,1]$:

$$[T_{N,x}(k_1,k_2)](\tau) :=$$

$$\sqrt{\frac{x-1}{N}}\,k_1\left(\frac{N\tau}{x-1}\right)1_{(N\tau\le x-1)} + \sqrt{\frac{N-x}{N}}\,k_2\left(\frac{N\tau-x}{N-x}\right)1_{(N\tau\ge x)}.$$

By the Markov property and the Brownian scaling of $(e_\tau)_{\tau\in[0,1]}$ we have:

$$\Sigma_h = \int_0^1 \frac{q^{1/2}\,h(r)}{\sqrt{2\pi r^3(1-r)^3}}\,T^*_r[m\otimes m]\,dr \tag{20}$$

where $T_r: K\times K\mapsto K$ and for $\tau\in[0,1]$:

$$[T_r(k_1,k_2)](\tau) := \sqrt{r}\,k_1\left(\frac{\tau}{r}\right)1_{(\tau\le r)} + \sqrt{1-r}\,k_2\left(\frac{\tau-r}{1-r}\right)1_{(r<\tau\le 1)}.$$

By Lemma 2 for all $r\in(0,1)$: $T^*_{N,\lfloor Nr\rfloor}\left[m_{\lfloor Nr\rfloor-1}\otimes m_{N-\lfloor Nr\rfloor}\right]$ tends to $T^*_r[m\otimes m]$ as $N\to\infty$. Therefore (14) follows from (19), (20) and the following

Lemma 3. *For all* $\delta\in(0,1/2)$*:*

$$\lim_{N\to\infty} N^{3/2}\,\frac{Z_{\lfloor Nr\rfloor}\,Z_{N-\lfloor Nr\rfloor}}{Z_N} = \frac{q^{1/2}}{\sqrt{2\pi r^3(1-r)^3}},$$

uniformly in $r\in[\delta,1-\delta]$.

Lemma 3 is proven at the end of the section. By (12), (14) and Lemma 2, now (13) holds for all $f \in C_b^1(H)$. In order to get convergence for all $f \in C_b(H)$, a uniform estimate on β_h^N is enough. This is the content of the next Lemma, proved at the end of the section.

Lemma 4. *For all $h \in C_c^2(0,1)$ there exists $C_h < \infty$ such that:*

$$\int_{K_N} |\beta_h^N|^2 \, dm_N \;\le\; C_h, \qquad \forall\, N \in \mathbb{N}.$$

For all $f \in C_b(H)$, $h \in C_c^2(0,1)$ and $N \in \mathbb{N}$ we set now:

$$T_N(f) \;:=\; \int_{K_N} \beta_h^N \, f \, dm_N \;+\; \int_{K_N} f \, d\Sigma_h^N,$$

$$T(f) \;:=\; \int_K \beta_h \, f \, dm \;+\; \int_K f \, d\Sigma_h.$$

If we prove that $T_N(f)$ converges to $T(f)$ for all $f \in C_b(H)$ as $N \to \infty$, then by (14) also (13) is proved. By the IbPF's (15) and (12) we have:

$$T_N(f) \;=\; -\int_{K_N} \partial_{(\Pi_N h)} f \, dm_N, \quad T(f) \;=\; -\int_K \partial_h f \, dm, \qquad \forall f \in C_b^1(H).$$

Therefore formula (13) holds for all $f \in C_b^1(H)$ by Lemma 2 and (14). In [10] it is proved that $C_b^1(H)$ is dense in $C_b(H)$ in the uniform convergence. By Lemmas 3 and 4 we have:

$$\sup_N \|T_N\| \;<\; \infty,$$

where $\|T_N\| := \sup\{|T_N(f)| \,:\, f \in C_b(H),\ \sup_H |f| \le 1\}$. Then the family of linear bounded functionals $(T_N)_N$ on $C_b(H)$ is equicontinuous and converges to the continuous functional T on a dense subset, so that it converges to T on the whole $C_b(H)$ and (13) is proved. This completes the proof of Theorem 1. □

We recall now the notation $S_n := X_1 + \cdots + X_n$, $(X_i)_i$ i.i.d. with $X_i \sim \exp(-V)dr$, $P_N :=$ law of $(S_1, \ldots, S_N)$ conditioned on $\{S_{N+1} = 0\}$.

Proof of Lemma 2. We sketch a proof which has been suggested by Jean Bertoin. We denote by $(b_\tau)_{\tau\in[0,1]}$ a standard Brownian bridge between 0 and 0 over [0, 1]. By Donsker's Theorem and the Local Limit Theorem for the density of $S_N/\sqrt{N}$, see [12, Th. 15 p. 206], the law of Λ_N under P_N converges weakly in the Skorohod topology to the law of $q^{-1/2}b$.

Under P_N the random variables $\{\phi_{i+1} - \phi_i,\ i = 0, \ldots, N\}$, with $\phi_0 := \phi_{N+1} := 0$, are exchangeable. Moreover P_N-a.s. $\phi_i \ne \phi_j$ for $i \ne j$, so that

$$T(\phi) := i \in \{0, \ldots, N\} \iff \phi_i \le \phi_j, \quad \forall j = 1, \ldots, N$$

is well defined. Moreover $\sum_{i=0}^N (\phi_{i+1} - \phi_i) = 0$. Therefore we obtain:

$$P_N\left((\phi_{i\oplus_N T}-\phi_T)_{i\in\Gamma_N}\in A,\ T=j\right)=\ P_N(\phi\in A,\ T=0)=\frac{\mu_N(A)}{N+1},\quad (21)$$

where $\oplus_N$ denotes the sum mod $N+1$ and $A\subseteq\mathbb{R}^{\Gamma_N}$ is Borel. Therefore T and $(\phi_{i\oplus_N T}-\phi_T)_i$ are independent under P_N, T is uniformly distributed on $\{0,\dots,N\}$ and $(\phi_{i\oplus_N T}-\phi_T)_i$ has law μ_N. We define now measurable maps:

$$\zeta : D([0,1])\mapsto[0,1],\qquad \zeta(\omega) := \inf\left\{s\in[0,1]:\ \omega(s) = \inf_{[0,1]}\omega\right\},$$

$$\Delta : D([0,1])\mapsto D([0,1]),\qquad \Delta_\tau(\omega) := \omega_{\tau\oplus\zeta}-\omega_\zeta,\quad \tau\in[0,1],$$

where $\oplus$ denotes the sum modulo 1 and $\inf\emptyset := 0$. By (21), m_N is the law of $\Delta\circ\Lambda_N$ under P_N. Moreover, by [15] m is the law of $\Delta(q^{-1/2}b)$.

We denote by D_ζ, respectively D_Δ, the set of $\omega\in D([0,1])$ such that ζ, resp. Δ, is discontinuous at ω in the Skorohod topology. Since m-a.e. ω is strictly positive over $(0,1)$, it is easy to see that $\mathbb{P}(q^{-1/2}b\in D_\zeta)=\mathbb{P}(q^{-1/2}b\in D_\Delta)=0$. Since Λ_N under P_N converges in law to $q^{-1/2}b$, by the Mapping Theorem (see e.g. Theorem 2.7 in [1]) we obtain the thesis. □

Proof of Lemma 3. Notice first that:

$$P_N = \frac{1}{z_N}e^{-H_{N,0}(\phi)}\prod_{x\in\Gamma_N}d\phi(x),\quad z_N := \int_{\mathbb{R}^{\Gamma_N}}e^{-H_{N,0}(\phi)}\prod_{x\in\Gamma_N}d\phi(x).$$

Since $\mu_N = P_N(\,\cdot\,|\,\Omega_N^+)$ and $P_N(\Omega_N^+)=P_N(T=0)=(N+1)^{-1}$ by (21), then:

$$Z_N = \int_{\Omega_N^+}e^{-H_{N,0}(\phi)}\prod_{x\in\Gamma_N}d\phi(x) = z_N\,P_N(\Omega_N^+) = \frac{z_N}{N+1}.$$

The thesis follows if we prove that:

$$z_N = \left(\frac{q}{2\pi N}\right)^{1/2} + o(N^{-1/2}).$$

By the change of variable $y_i := \phi_i-\phi_{i-1}$, $i=1,\dots,N$, we obtain:

$$z_N = \mathbb{E}\left[\exp(-V(S_N))\right] = \int_{\mathbb{R}}\exp\left(-V\left(a\sqrt{N/q}\right)\right)\,p_N(a)\,da,$$

where p_N denotes the distribution density of $\sqrt{q/N}\,S_N$ under $\mathbb{P}$. Note that we have the following expansion, uniformly in $a\in\mathbb{R}$

$$p_N(a) = q_0(a) + \frac{1}{N^{1/2}}\,q_1(a) + o(N^{-1/2}),$$

where q_0 is the density of $\mathcal{N}(0,1)$, $q_1 = c_1\,q_0H_3$, c_1 is a constant and H_3 is the Chebyshev-Hermite polynomial of degree 3, cf. pp. 138 and 206 of [12]. Then, setting:

$$I_i := N^{-i/2}\int_{[-1,1]}\exp\left(-V\left(a\sqrt{N/q}\right)\right)\,q_i(a)\,da,\qquad i=0,1,$$

we obtain by (V3) above: $|z_N - I_0 - I_1| = o(N^{-1/2})$, $I_1 = O(N^{-1})$,

$$I_0 = \left(\frac{q}{2\pi N}\right)^{1/2} + o(N^{-1/2}). \qquad \square$$

We recall now the random walk representation, introduced in [4] and extended to reflected systems in [5]. Let E denote the set $D([0,\infty);\Gamma_N)$ of Γ_N-valued càdlàg functions and ξ the coordinate process in E. Let $\phi = \phi^U$ be the unique solution of (6) with $\phi_0 \in \Omega_N^+$ and set $\phi_t(x) := 0$ for all $t \geq 0$, $x \in \mathbb{Z}\backslash\Gamma_N$. We denote by $\mathbf{P}_{x_0}^{\phi}$ the law of the continuous time random walk $(\xi_t)_{t\geq 0}$ in $\mathbb{Z}$ which jumps from x to y at time t with rate

$$a_t(x,y) := V''(\phi_t(x) - \phi_t(y))\, 1_{(|y-x|=1)},$$

and such that $\mathbf{P}_{x_0}^{\phi}(\xi_0 = x_0) = 1$. We set:

$$\tau := \left(\inf_{x\in\Gamma_N} \inf\{t>0 : \phi_t(x) = 0, \xi_t = x\}\right) \wedge \inf\{t>0 : \xi_t \in \partial\Gamma_N\}. \tag{22}$$

For $F \in C_b^1(\mathbb{R}^{\Gamma_N})$ we set $\partial F(x,\phi) := \partial F/\partial\phi(x)$ for $x \in \Gamma_N$. Then it has been proved in Theorem 2 of [5] that for all $F \in C_b^1(\mathbb{R}^{\Gamma_N})$:

$$\frac{\partial}{\partial\phi_0(y)}\mathbb{E}\left[F(\phi^U(t,\phi_0))\right] = \mathbb{E}\left[\mathbf{E}_y^{\phi}\, \partial F\left(\xi_t, \phi^U(t,\phi_0)\right)\, 1_{(t<\tau)}\right].$$

Consider now the case $U = 0$. We denote by $\hat{C}$ as the space of all $g : \Gamma_N \times \Omega_N^+$ such that $g(x,\cdot) \in C_b^1(\Omega_N^+)$ for all $x \in \Gamma_N$. Then the homogeneous Markov process (ϕ^0, ξ) is associated with the closure of the pre-Dirichlet Form:

$$\mathcal{I}_N(g,g) := \frac{1}{2}\int \sum_{x=0}^{N} \left\{ \|\nabla g(x,\cdot)\|^2 + \left[V''(D_N\phi)\,|D_N g(\cdot,\phi)|^2\right](x)\right\} d\mu_N,$$

where $g \in \hat{C}$, $g(0,\cdot) := g(N+1,\cdot) := 0$, $D_N : \mathbb{R}^{\Gamma_N} \mapsto \mathbb{R}^{\Gamma_N}$, $D_N f(x) := f(x+1) - f(x)$, $x = 0,\ldots,N$, with $f(0) := f(N+1) := 0$. Moreover, arguing like in Lemma 2.2 of [4] we obtain:

$$\begin{aligned}&\int F^2\, d\mu_N - \left(\int F\, d\mu_N\right)^2 \\ &= \sum_{x\in\Gamma_N} \int \partial F(x,\phi)\, \mathbb{E}\left[\mathbf{E}_x^{\phi} \int_0^{\tau} \partial F(\xi_s, \phi^0(s,\phi))\, ds\right] \mu_N(d\phi). \end{aligned}\tag{23}$$

Proof of Lemma 4. We follow the proof of Lemma 5.6 in [9]. We set $D_N^* : \mathbb{R}^{\Gamma_N} \mapsto \mathbb{R}^{\Gamma_N}$, $D_N^* f(x) := f(x-1) - f(x)$, $x = 1,\ldots,N+1$, with $f(0) := f(N+1) := 0$. Moreover we denote by $(-D_N^* D_N)^{-1}$ the inverse of $-D_N^* D_N$ with Dirichlet boundary conditions on $\partial\Gamma_N$. We set now

$$G(\phi) := \sum_{x=1}^{N+1} \left[\overline{h}(x) - \overline{h}(x-1)\right] V'(\phi(x) - \phi(x-1)),$$

where $\overline{h}(x) := N^{3/2}\langle h, 1_{I(x)}\rangle$. Then:

$$G(\phi) \; = \; \sum_{x=1}^{N+1} \left\{ \left(D_N^* \overline{h}\right) \cdot V' \left(D_N^* \phi\right) \right\} (x),$$

$$\partial G(x,\phi) \; = \; - D_N \left\{ (D_N^* \overline{h}) \cdot V''(D_N^* \phi) \right\} (x), \qquad x \in \Gamma_N.$$

By (23) and the variational characterization of the Dirichlet Form $\mathcal{I}_N$:

$$\begin{aligned}
&\int G^2 \, d\mu_N - \left[\int G \, d\mu_N \right]^2 \\
&\le \sup_{g \in \hat{C}} \left\{ 2 \sum_{x \in \Gamma_N} \int \left[g(x,\phi) \, \partial G(x,\phi) - \mathcal{I}_N(g,g) \right] \mu_N(d\phi) \right\} \\
&\le \sup_{g \in \hat{C}} \left\{ 2 \sum_{x \in \Gamma_N} \int \left[g(x,\phi) \, \partial G(x,\phi) - \frac{C_-}{2} \left| D_N g(\cdot,\phi) \right|^2 (x) \right] \mu_N(d\phi) \right\} \\
&\le \frac{2}{C_-} \int \sum_{x \in \Gamma_N} \partial G(x,\phi) \left[(-D_N^* D_N)^{-1} \partial G(\cdot,\phi) \right] (x) \, \mu_N(d\phi) \\
&= \frac{2}{C_-} \sum_{x=1}^{N+1} \left| \overline{h}(x) - \overline{h}(x-1) \right|^2 \int \left[V''(\phi(x) - \phi(x-1)) \right]^2 \mu_N(d\phi) \\
&\le \frac{2C_+^2}{C_-} \sum_{x=1}^{N+1} \left| \overline{h}(x) - \overline{h}(x-1) \right|^2 \; \le \; \frac{2C_+^2}{C_-} \, \|h'\|_\infty.
\end{aligned}$$

We estimate now the average of G w.r.t. μ_N. By the definition of β_h^N, $G = \beta_h^N \circ \Lambda_N$. Setting $F \equiv 1$ and $a(x) := \overline{h}(x)$ in (16), by Lemma 3:

$$\int G \, d\mu_N \; = \; \sum_{x \in \Gamma_N} N^{3/2} \, \langle h, I_{(x)} \rangle \, \frac{Z_{x-1} \, Z_{N-x}}{Z_N} \; \stackrel{N \to \infty}{\longrightarrow} \; \int_0^1 \frac{q^{1/2} \, h(r)}{\sqrt{2\pi r^3 (1-r)^3}} \, dr,$$

and the thesis follows, since h has compact support in (0, 1). □

4. Equicontinuity of resolvents

We denote by $(R_\lambda^{N,U})_{\lambda > 0}$ the resolvent of $\mathcal{E}^{N,U}$:

$$R_\lambda^{N,U} f(k) \; := \; \int_0^\infty e^{-\lambda t} \, \mathbb{E} \left[f \left(\Phi_N^U(t,k) \right) \right] dt, \qquad f \in L^2(m_N^U), \quad k \in K_N.$$

For all $J \subset H$ and $f : J \mapsto \mathbb{R}$ we set:

$$[f]_{\mathrm{Lip(J)}} \; := \; \sup_{h,k \in J, h \neq k} \frac{|f(h) - f(k)|}{\|h - k\|}.$$

This section is devoted to the proof of the following:

Lemma 5. *There exists a constant* $C = C(c_-, c_+) > 0$ *such that for all* $N \in \mathbb{N}$, $\lambda > 0$ *and* $f \in C_b(K_N)$*:*

$$[R_\lambda^{N,U} f]_{\mathrm{Lip(K_N)}} \leq C\,\lambda^{-1/4}\,\|f\|_\infty, \qquad \forall\,\lambda > 0. \tag{24}$$

Proof. It has been proved in Theorem 2 of [5] that for all $F \in C_b(\mathbb{R}^{\Gamma_N})$:

$$\frac{\partial}{\partial\phi_0(y)}\mathbb{E}\left[F(\phi^U(t,\phi_0))\right] = \frac{1}{t}\,\mathbb{E}\left[F(\phi^U(t,\phi_0))\sum_{x\in\Gamma_N}\int_0^t \eta_s(x,y)\,dw_s(x)\right],$$

where $\eta_s(x,y) := \mathbf{P}_y^\phi(\xi_s = x, s < \tau)$, see (22) above. This is a Bismut-Elworthy's formula, i.e. a probabilistic representation for the gradient of the transition semigroup of the Markov process ϕ^U, depending on the function F but not on its derivatives. Then:

$$\left|\frac{\partial}{\partial\phi_0(y)}\mathbb{E}\left[F(\phi^U(t,\phi_0))\right]\right|^2 \leq \frac{\|F\|_\infty^2}{t^2}\int_0^t \sum_{x\in\Gamma_N}\mathbb{E}\left[|\eta_s(x,y)|^2\right]ds.$$

By (1.2) in [2] there exist $c_1, c_2 > 0$, depending only on (c_-, c_+), such that:

$$0 \leq \eta_s(x,y) \leq \mathbf{P}_y^\phi(\xi_s = x) \leq c_1\,p^*(c_2 s, y, x),$$

where p^* is the transition probability of the continuous time random walk in $\mathbb{Z}$ with jump rate from y to x equal to $1_{(|x-y|=1)}$, $x, y \in \mathbb{Z}$. Then:

$$\sum_{x\in\Gamma_N}|\eta_s(x,y)|^2 \leq \sum_{x\in\Gamma_N} c_1^2\left[p^*(c_2 s, y, x)\right]^2 = c_1^2\,p^*(2c_2 s, y),$$

by the Markov property and the symmetry $p^*(s,y,x) = p^*(s,x,y)$. Now by Lemma B.2 in [9]: $p^*(s,y) \leq c_3\,s^{-1/2}$ for some $c_3 = c_3(c_-, c_+) > 0$. Then:

$$\sum_{y\in\Gamma_N}\left|\frac{\partial}{\partial\phi_0(y)}\mathbb{E}\left[F(\phi^U(t,\phi_0))\right]\right|^2 \leq \frac{N(c_4\|F\|_\infty)^2}{t^{3/2}}, \qquad t > 0,$$

for some $c_4 = c_4(c_-, c_+) > 0$. In particular for all $\phi, \phi' \in K_N$:

$$\left|\mathbb{E}\left[F(\phi^U(t,\phi_0))\right] - \mathbb{E}\left[F(\phi^U(t,\phi_0'))\right]\right| \leq \frac{c_4 N^{1/2}\|F\|_\infty}{t^{3/4}}\,\|\phi_0 - \phi_0'\|_{\mathbb{R}^{\Gamma_N}}.$$

If now $F = f\circ\Lambda_N$, $k = \Lambda_N(\phi_0)$ and $k' = \Lambda_N(\phi_0')$, then by (10):

$$\left|\mathbb{E}\left[f(\Phi_N^U(t,k))\right] - \mathbb{E}\left[f(\Phi_N^U(t,k'))\right]\right| \leq \frac{c_4\|f\|_\infty}{t^{3/4}}\,\|k-k'\|_H\,\frac{N^{\frac{1}{2}}\cdot N}{N^{2\cdot\frac{3}{4}}}, \tag{25}$$

which yields the thesis after taking the Laplace transform in time. □

5. Convergence of the reflected interface

In this section we prove convergence of the fluctuation field Φ_N^U. We introduce the probability measure on K:

$$m^U := \frac{1}{Z^U} e^{-\langle U(k),1\rangle} m(dk), \qquad Z^U := \int_K e^{-\langle U(k),1\rangle} m(dk).$$

We recall the following result, proved in [16]:

Proposition 1. *Denote by (v, ζ) be the solution of the SPDE with reflection:*

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{q}{2}\dfrac{\partial^2 v}{\partial \theta^2} - \dfrac{1}{2} U'(v) + \dfrac{\partial^2 W}{\partial t \partial \theta} + \zeta \\ v(0, \cdot) = k \in K, \ v(t, 0) = v(t, 1) = 0 \\ v \geq 0, \ d\zeta \geq 0, \ \int v \, d\zeta = 0, \end{cases} \tag{26}$$

where $(W(t, \theta))_{t\geq 0, \theta\in[0,1]}$ is a Brownian sheet, $v := (0, \infty) \times [0, 1] \mapsto \mathbb{R}$ is continuous and ζ is a radon positive measure on $(0, \infty) \times (0, 1)$: see [11]. We write $v^U(t, k) := v(t, \cdot)$. Then:

1. *The Markov process $(v^U(t, k))_{t\geq 0, k\in K}$ is the diffusion generated by the symmetric Dirichlet Form $(\mathcal{E}^U, D(\mathcal{E}^U))$ in $L^2(m^U)$, closure of:*

$$\mathrm{Exp}(H) \ni \varphi, \psi \mapsto \frac{1}{2}\int_K \langle \nabla\varphi, \nabla\psi\rangle \, dm^U.$$

2. *m^U is the unique invariant probability measure of $(v^U(t, k))_{t\geq 0, k\in K}$. We denote by $(v^U(t))_{t\geq 0}$ the unique stationary solution of* (26).
3. *The space* $\mathrm{Lip}(K)$ *of Lipschitz functions on K is contained in $D(\mathcal{E}^U)$.*

We introduce the space $H^{-\gamma}(0, 1)$, $\gamma > 0$, completion of H w.r.t. the norm:

$$\|f\|^2_{-\gamma} := \sum_{n=1}^{\infty} n^{\gamma} \, |\langle f, e_n\rangle|^2$$

where $e_n(\theta) := \sqrt{2}\sin(n\pi\theta)$, $\theta \in [0, 1]$. Then the main result of this section is the following:

Theorem 2.

1. *For all $T > 0$ and $\gamma > 1$, $(\Phi_N^U(t))_{t\in[0,T]}$ weakly converges to $(v^U(t))_{t\in[0,T]}$ in $C([0, T]; H^{-\gamma}(0, 1))$ as $N \to \infty$.*
2. *Let γ_N be any K_N-valued r.v. independent of Φ_N^U and γ any K-valued r.v. independent of v^U, such that $\gamma_N \to \gamma$ in distribution in $L^2(0, 1)$. Then for all $f \in C_b(K^n)$, $0 < t_1 < \cdots < t_n$:*

$$\mathbb{E}\Big[f(\Phi_N^U(t_i, \gamma_N), i = 1, \ldots, n)\Big] \overset{N\to\infty}{\longrightarrow} \mathbb{E}\Big[f(v^U(t_i, \gamma), i = 1, \ldots, n)\Big].$$

Tightness of $(\Phi_N^U)_N$ is proved in Lemma 7. The identification of limit points is based on the Markov property and on the identification of the one time marginals. To this aim, the main technical step is the following:

Lemma 6. *For all $k \in K$, $f \in C_b(K)$ and $\lambda > 0$ we have:*

$$\int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[f(\Phi_N^U(t, \Pi_N k))\right] dt \overset{N\to\infty}{\longrightarrow} \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[f(v^U(t,k))\right] dt. \quad (27)$$

In the following proofs we use a number of times without further mention the following easily proven fact: if E is a Polish space, $(M_n : n \in \mathbb{N} \cup \{\infty\})$ is a sequence of probability measures on E, such that M_n converges weakly to M_∞, and $(f_n : n \in \mathbb{N} \cup \{\infty\})$ is an equi-bounded and equi-continuous sequence of functions on E, such that f_n converges pointwise to f_∞, then $\int_E f_n\, dM_n \to \int_E f_\infty\, dM_\infty$ as $n \to \infty$.

Proof of Lemma 6. First we outline the proof. Let us fix $\lambda > 0$ and set:

$$F_N(k) := \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[f(\Phi_N^U(t, \Pi_N k))\right] dt, \quad k \in K,$$

$$\mathcal{E}_\lambda^{N,U}(\varphi, \psi) := \lambda \int_{K_N} \varphi\, \psi\, dm_N^U + \mathcal{E}^{N,U}(\varphi, \psi), \qquad \varphi, \psi \in D(\mathcal{E}^{N,U}).$$

We recall that F_N is the unique function in $D(\mathcal{E}^{N,U})$ such that:

$$\mathcal{E}_\lambda^{N,U}(F_N, g) = \int_{K_N} f\, g\, dm_N^U, \qquad \forall g \in D(\mathcal{E}^{N,U}). \quad (28)$$

Suppose that F_N converges pointwise to $F \in C_b(K)$ and:

$$\mathcal{E}_\lambda^{N,U}(F_N, g) \overset{N\to\infty}{\longrightarrow} \mathcal{E}_\lambda^U(F, g), \qquad \forall\, g \in \mathrm{Exp}(H). \quad (29)$$

Then, by (28) and the weak convergence of m_N^U to m^U:

$$\lambda \int_K F\, g\, dm^U + \mathcal{E}^U(F, g) = \int_K f\, g\, dm^U, \qquad \forall g \in \mathrm{Exp}(H). \quad (30)$$

By point 1 of Proposition 1, $\mathrm{Exp}(H)$ is a core of $D(\mathcal{E}^U)$, so that (30) implies:

$$F(k) = \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left[f(v^U(t,k))\right] dt, \quad m^U - \text{a.e. } k \in K, \quad (31)$$

and, by continuity, pointwise on K, which concludes the proof.

The problem in the proof of (29), is that the l.h.s. involves the gradient of F_N. On the other hand, we can integrate by parts and shift all derivatives on the test function g: in this way we rewrite (28) as an equality where F_N appears but none of its derivatives does, and we can apply Theorem 1 and Lemma 5. To this aim, we set for all $h \in C_c^2(0,1)$:

$$\Sigma_h^U(dk) := \frac{1}{Z^U} e^{-\langle U(k),1\rangle} \Sigma_h(dk) + \langle U'(k), h\rangle m^U(dk),$$

$$\Sigma_h^{N,U}(dk) := \frac{1}{Z_N^U} e^{-\langle U(k),1\rangle} \Sigma_h^N(dk) + \langle U'(k), h\rangle m_N^U(dk).$$

Then by (15) and (12) we have IbPF's for m^U and m_N^U: for all $f \in C_b^1(H)$ and $h \in C_c^2(0,1)$

$$\int_K \partial_h f \, dm^U = -\int_K \beta_h \, f \, dm^U - \int_K f \, d\Sigma_h^U, \tag{32}$$

$$\int_{K_N} \partial_{(\Pi_N h)} f \, dm_N^U = -\int_{K_N} \beta_h^N \, f \, dm_N^U - \int_{K_N} f \, d\Sigma_h^{N,U}. \tag{33}$$

Step 1. By Lemmas 2-3 and by (14), for all $h \in C_c^2(0,1)$ there exists a sequence of compact sets $(J_n)_n$ in K such that:

$$\lim_{n\to\infty} \sup_N \left[\int_{K\setminus J_n} \left(1 + \left|\beta_h^N\right|\right) dm_N + \Sigma_{|h|}^N (K\setminus J_n) \right] = 0. \tag{34}$$

Set $J := \cup_n J_n$. Since $m(J) = 1$ and m is the law of a Bessel Bridge, then J is dense in K. By Lemma 5: $\sup_N \|F_N\|_\infty + [F_N]_{\mathrm{Lip}(K)} < \infty$. Let $(N_j)_j$ be any sequence in $\mathbb{N}$. With a diagonal procedure, by Ascoli-Arzelà's Theorem we can find a subsequence $(N_{j_i})_i$ and a function $F : J \mapsto \mathbb{R}$ such that:

$$\lim_i \sup_{J_n} |F_{N_{j_i}} - F| = 0, \qquad \forall n. \tag{35}$$

Moreover, F can be extended uniquely to a bounded Lipschitz function on K which we still denote by F and $F_N \to F$ pointwise on K.

Step 2. For all $h \in C_c^2(0,1)$, let $g_h : K \mapsto \mathbb{C}$, $g_h(k) := \exp(i\langle h, k\rangle)$, where $i \in \mathbb{C}$ and $i^2 = -1$. Notice that for all $k \in H_N$: $g_h(k) = g_{(\Pi_N h)}(k)$. Moreover $\nabla g_h = i\, h\, g_h$. The IbPF (33) yields:

$$\mathcal{E}_\lambda^{N,U}(F_N, g_h) = \frac{1}{2}\int_{K_N} F_N \left(c_{\lambda,h}^N - i\beta_h^N\right) g_h \, dm_N^U - \frac{1}{2}\int_{K_N} F_N \, i \, g_h \, d\Sigma_h^{N,U},$$

where $c_{\lambda,h}^N := 2\lambda + \|\Pi_N h\|^2$. By (34) and (35) we can prove that along the subsequence $(N_{j_i})_i$ we have convergence of the r.h.s. of the last formula to:

$$\frac{1}{2}\int_K F\left(c_{\lambda,h} - i\,\beta_h\right) g_h \, dm^U - \frac{1}{2}\int_K F \, i \, g_h \, d\Sigma_h^U = \mathcal{E}_\lambda^U(F, g_h)$$

where $c_{\lambda,h} := 2\lambda + \|h\|^2$ and in the last equality we have applied the IbPF (32). Then, we obtain (29) along the subsequence $(N_{j_i})_i$ and, arguing as above, F is equal to the r.h.s. of (27). Since this is true for any subsequence $(N_j)_j$, the thesis is proven. □

We recall that the Hilbert-Schmidt norm of the inclusion $H \to H^{-1}(0,1)$ is finite, and the inclusion $H^{-1}(0,1) \to H^{-\gamma}(0,1)$ is compact for all $\gamma > 1$. Then we have:

Lemma 7. *For all $p > 1$ there exists $C_p \in (0,\infty)$, independent of U, such that for all $N \in \mathbb{N}$:*

$$\left(\mathbb{E}\left[\left\|\Phi_N^U(t) - \Phi_N^U(s)\right\|_{H^{-1}(0,1)}^p\right]\right)^{\frac{1}{p}} \leq C_p\,|t-s|^{\frac{1}{2}}, \qquad t,s \geq 0. \tag{36}$$

In particular, for all $\gamma > 1$, $(P_N^U)_N$ is tight in $C([0,T];H^{-\gamma}(0,1))$.

Proof. Fix $N \in \mathbb{N}$ and $T > 0$. By the Lyons-Zheng decomposition, see e.g. [7, Th. 5.7.1], we have for $t \in [0,T]$:

$$\Phi_N^U(t) - \Phi_N^U(0) = \frac{1}{2}M_t^1 - \frac{1}{2}\left(M_T^2 - M_{T-t}^2\right), \quad P_N^U - \text{a.s.}$$

where M^i is a H_N-valued $(\mathcal{F}_t^i)_t$-martingale, $i = 1,2$, $\mathcal{F}_t^1 := \sigma(\Phi_N^U(s), s \leq t)$ and $\mathcal{F}_t^2 := \sigma(\Phi_N^U(T-s), s \leq t)$. Moreover, the quadratic variations are: $\langle M^i\rangle_t = t \cdot I_N$, where I_N is the identity matrix in H_N. By the Burkholder-Davis-Gundy inequality we can find $c_p \in (0,\infty)$ for all $p > 1$ such that:

$$\left(\mathbb{E}\left[\left\|\Phi_N^U(t) - \Phi_N^U(s)\right\|_{H^{-1}(0,1)}^p\right]\right)^{\frac{1}{p}} \leq c_p\,N_{0\to-1}\,|t-s|^{\frac{1}{2}}, \quad t,s \in [0,T],$$

where $N_{0\to-1}$ is the Hilbert-Schmidt norm of the inclusion $H \to H^{-1}(0,1)$. Since the law m_N^U of $\Phi_N^U(0)$ converges by Lemma 2, tightness of Φ_N^U follows e.g. by Theorem 7.2 in Chap. 3 of [6]. □

Proof of Theorem 2. We prove first point 1. For all $T \geq 0$, we denote by P_N^U the law of $(\Phi_N^U(t))_{t\in[0,T]}$ on $C([0,T];K)$. By Lemma 7 we only have to identify any limit point of P_N^U with the law $\mathbb{P}^U$ of $(v^U(t))_{t\in[0,T]}$. We set for all $t \geq 0$: $X_t : C([0,T];K) \mapsto K$, $X_t(\omega) := \omega(t)$. Let $f_1,\ldots,f_n \in C_b(K)$. Arguing by induction on n and using Lemmas 5 and 6 we obtain for all $\lambda_1,\ldots,\lambda_n \in (0,\infty)$:

$$\lim_{N\to\infty}\int m_N^U(dk)\,R_{\lambda_1}^{N,U}(f_1 R_{\lambda_2}^{N,U}(f_2\cdots R_{\lambda_n}^{N,U}(f_n))\cdots)(\Pi_N k)$$

$$= \int m^U(dk)\,R_{\lambda_1}^U(f_1 R_{\lambda_2}^U(f_2\cdots R_{\lambda_n}^U(f_n))\cdots)(k), \qquad \forall k \in K,$$

where $(R_\lambda^U)_{\lambda>0}$ is the resolvent of $\mathcal{E}^U$. Let now $(N_i)_i$ be a subsequence such that $P_{N_i}^U$ has a weak limit P. Then:

$$\int_{(0,\infty)^n} e^{-\sum_{j=1}^n \lambda_j t_j}\int f_0(X_0)\prod_{j=1}^n f_j(X_{t_1+\cdots+t_j})\,dP\,dt_1\cdots dt_n$$

$$= \int_{(0,\infty)^n} e^{-\sum_{j=1}^n \lambda_j t_j}\int f_0(X_0)\prod_{j=1}^n f_j(X_{t_1+\cdots+t_j})\,d\mathbb{P}^U\,dt_1\cdots dt_n,$$

and by the uniqueness of the Laplace transform the thesis follows.

Let us now prove point 2. First notice that by (7), a.s.

$$\|\Phi_N^U(t,k) - \Phi_N^U(t,k')\| \le e^{-c_- t/2} \|k - k'\|, \qquad t \ge 0, \; k, k' \in K_N. \tag{37}$$

By (37) and by the independence of γ_N and Φ_N^U it is enough to consider deterministic $\gamma \equiv k \in K$ and $\gamma_N \equiv \Pi_N k$. Let $f \in C_b(K)$. We consider the spectral measure ν_N^f of the generator of $\mathcal{E}^{N,U}$ associated with (the restriction to K_N of) f:

$$\int f \, P_t^{N,U} f \, dm_N^U = \int_{-\infty}^0 e^{tx} \, \nu_N^f(dx), \qquad t \ge 0,$$

where $(P_t^{N,U})_{t \ge 0}$ denotes the transition semigroup of Φ_N^U. The convergence of the resolvents as $N \to \infty$ and Lemma 5 imply that for all $\lambda > 0$, $\ell \in \mathbb{N}$:

$$\int_{-\infty}^0 \frac{1}{(\lambda - x)^\ell} \, \nu_N^f(dx) = \int f \left(R_\lambda^{N,U}\right)^\ell f \, dm_N^U$$

$$\overset{N \to \infty}{\longrightarrow} \int f \left(R_\lambda^U\right)^\ell f \, dm^U = \int_{-\infty}^0 \frac{1}{(\lambda - x)^\ell} \, \nu^f(dx),$$

where ν^f is the spectral measure of the generator of $\mathcal{E}^U$ associated with f. By the Stone-Weierstrass theorem, the vector space spanned by the set of functions $\{x \mapsto (\lambda - x)^{-\ell}, \lambda > 0, \ell \in \mathbb{N}\}$ is dense in the set $C_0((-\infty, 0])$ of continuous functions on $(-\infty, 0]$ vanishing at $-\infty$. Therefore, using also (25) and the polarization identity, we have

$$\int g \, P_t^{N,U} f \, dm_N^U \overset{N \to \infty}{\longrightarrow} \int g \, P_t^U f \, dm^U$$

where $(P_t^U)_{t \ge 0}$ denotes the transition semigroup of v^U and $g \in C_b(K)$. Arguing as in the proof of Lemma 6 we can extract from $([P_t^{N,U} f] \circ \Pi_N)_N$ subsequences converging pointwise and, using the last formula, identify all possible limits with $P_t^U f$. Using (25) and the Markov property we obtain the thesis. □

6. Interface with repulsion

In this section we introduce the interface model with repulsion from the wall, solution of (2) above, and prove convergence of the associated fluctuation field. The proof is obtained considering first the reflected interface ϕ^{U_ϵ}, where for $\epsilon > 0$, $c > 0$ and $\alpha > 0$ we set:

$$U_\epsilon(r) := c\,(\epsilon + r)^{-\alpha}, \quad r \ge 0, \qquad U_0(r) := c\,r^{-\alpha}, \quad r > 0.$$

Since U_ϵ is convex, bounded, non-negative and non-increasing on $[0, \infty)$, Theorem 2 can be applied to ϕ^{U_ϵ}. Moreover, $U_\epsilon(r) \uparrow U_0(r)$ for all $r > 0$ as $\epsilon \downarrow 0$. Using monotonicity arguments analogous to those of [8] and [17] we extend Theorem 1 to the interface with repulsion.

Since $U_0 \geq 0$ the following probability measure is well defined:

$$\mu_{N,a}^{c,\alpha}(d\phi) := \frac{1}{Z_{N,a}^{c,\alpha}} \exp\left\{-\frac{1}{N}\sum_{x\in\Gamma_N} U_0\left(N^{-1/2}\,\phi(x)\right)\right\} \mu_{N,a}(d\phi), \quad a \geq 0.$$

Lemma 8. *Let $c, \alpha > 0$, $a \geq 0$.*

1. *For all $\phi_0 \in \Omega_N^+$ there exists a unique process $(\phi_t)_{t\geq 0}$ in Ω_N^+, such that a.s. for all $t > 0$: $\phi_t(x) > 0$, $\int_0^t [\phi_s(x)]^{-1-\alpha} ds < \infty$,*

$$\phi_t(x) = \phi_0(x) - \frac{1}{2}\int_0^t \{V'(\phi_s(x) - \phi_s(x-1)) - V'(\phi_s(x+1) - \phi_s(x))\} ds$$

$$+ N^{\frac{\alpha-2}{2}} \frac{c\alpha}{2} \int_0^t \frac{1}{[\phi_s(x)]^{\alpha+1}}\, ds + w_t(x), \qquad x \in \Gamma_N, \tag{38}$$

and $\phi_t(0) = \phi_t(N+1) = \sqrt{N}\,a$. We write $\phi_a^{c,\alpha}(t, \phi_0) := \phi_t$.

2. *$\mu_{N,a}^{c,\alpha}$ is the unique invariant probability measure of $\phi_a^{c,\alpha}$: we denote by $(\phi_a^{c,\alpha}(t))_{t\geq 0}$ the unique stationary solution of* (38).
3. *For $a = 0$ and for all $\phi_0 \in \Omega_N^+$ we have $\phi^{U_\epsilon}(t, \phi_0) \uparrow \phi_0^{c,\alpha}(t, \phi_0)$ as $\epsilon \downarrow 0$ a.s. and $(\phi_0^{c,\alpha}(t))_{t\geq 0}$ is the limit in law of $(\phi^{U_\epsilon}(t))_{t\geq 0}$.*

Proof. Pathwise uniqueness of solutions of (38) for any initial condition $\phi_0 \in \Omega_N^+$ follows arguing as in the proof of Lemma 1. Let now $C > 0$ be a constant such that

$$N^{\frac{\alpha-2}{2}} \frac{c\alpha}{2} \frac{1}{r^{1+\alpha}} \geq \frac{1}{r} - C, \qquad \forall r \in (0, \infty).$$

By the Girsanov Theorem, for all $x \in \Gamma_N$ there exists a solution $\rho(x)$ of

$$\rho_t(x) = \phi_0(x) + \int_0^t \frac{1}{\rho_s(x)}\, ds - C\,t + w_t(x), \quad t \geq 0,$$

and the law of $\rho(x)$ is absolutely continuous w.r.t. the law of a Bessel process of dimension 3, so that a.s. $\rho_t(x) > 0$ for all $t > 0$, see [13]. For $\epsilon > 0$ we introduce now $(z_t^\epsilon(x))_{t\geq 0}$, solution of the SDE with reflection:

$$z_t^\epsilon(x) = \phi_0(x) + N^{\frac{\alpha-2}{2}} \frac{c\alpha}{2} \int_0^t \frac{1}{[\epsilon + z_s^\epsilon(x)]^{1+\alpha}}\, ds + l_t^\epsilon(x) + w_t(x), \qquad t \geq 0.$$

By a monotonicity argument, we find that a.s. the maps $\epsilon \mapsto z_t^\epsilon(x)$, $\epsilon \mapsto \epsilon + z_t^\epsilon(x)$, $\epsilon \mapsto l_t^\epsilon(x)$ are monotone and bounded, and the limits $(z, l) := \lim_{\epsilon\downarrow 0}(z^\epsilon, l^\epsilon)$ satisfy:

$$z_t(x) = \phi_0(x) + N^{\frac{\alpha-2}{2}} \frac{c\alpha}{2} \int_0^t \frac{1}{[z_s(x)]^{1+\alpha}}\, ds + l_t(x) + w_t(x), \qquad t \geq 0.$$

Applying Itô's formula to $\gamma_s := \sum_x [(\rho_s(x) - z_s(x))^+]^2$, we find by the choice of C that a.s. $z \geq \rho$, so that $z_t(x) > 0$ for all $t > 0$ and in particular $l \equiv 0$. By the Girsanov Theorem we can now construct a solution of (38). □

We recall that b denotes a standard Brownian bridge over $[0, 1]$. The following result is proved in [17].

Proposition 2. *Let $a \geq 0$, $c > 0$, $\alpha > 0$, $k \in K$.*

1. *For $\alpha \geq 2$, there exists a unique non-negative u, continuous on $(0, \infty) \times [0, 1]$, with $u^{-1-\alpha} \in L^1_{loc}([0, \infty) \times (0, 1))$, such that*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{q}{2} \dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{c\alpha}{2} \dfrac{1}{u^{1+\alpha}} + \dfrac{\partial^2 W}{\partial t \partial \theta} \\ u(0, \cdot) = k, \ u(t, 0) = u(t, 1) = a \end{cases} \tag{39}$$

   We write $u^{c,\alpha}_a(t, k) := u(t, \cdot)$.

2. *For $0 < \alpha < 2$, there exists a unique (u, ζ), with u non-negative continuous on $(0, \infty) \times [0, 1]$, $u^{-1-\alpha} \in L^1_{loc}([0, \infty) \times (0, 1))$, ζ non-negative radon measure on $(0, \infty) \times (0, 1)$, such that*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{q}{2} \dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{c\alpha}{2} \dfrac{1}{u^{1+\alpha}} + \dfrac{\partial^2 W}{\partial t \partial \theta} + \zeta \\ u(0, \cdot) = k, \ u(t, 0) = u(t, 1) = a, \quad \displaystyle\int u \, d\zeta \ = \ 0 \end{cases} \tag{40}$$

   We write $u^{c,\alpha}_a(t, k) := u(t, \cdot)$. With positive probability, $\zeta([0, t] \times [0, 1]) > 0$ for some $t > 0$.

3. *For all $\alpha > 0$ there exists a unique invariant measure $m^{c,\alpha}_a$ for (39), resp. (40). We denote by $(u^{c,\alpha}_a(t))_{t \geq 0}$ the unique stationary solution of (39), resp. (40). If $a > 0$ then:*

$$m^{c,\alpha}_a(d\omega) \ = \ \frac{1}{Z^{c,\alpha}_a} \ \exp(-\langle U_0(\omega), 1\rangle) \, m^a(d\omega), \tag{41}$$

   where m^a is the law of $(a + q^{-1/2} b)$ conditioned on K. Moreover $m^{c,\alpha}_a \to m^{c,\alpha}_0$ and $m^{U_\epsilon} \to m^{c,\alpha}_0$ weakly in K as $a, \epsilon \to 0$.

We can finally state and prove the convergence result for the fluctuations of the interface with repulsion. We set for $t \geq 0$, $k \in K_N$ and $a \geq 0$:

$$\Phi^{c,\alpha}_{N,a}(t, k) \ := \ \Lambda_N(\phi^{c,\alpha}_a(N^2 t, \Lambda_N^{-1}(k))), \quad \Phi^{c,\alpha}_{N,a}(t) \ := \ \Lambda_N(\phi^{c,\alpha}_a(N^2 t)),$$

$$m^{c,\alpha}_{N,a} \ := \ \Lambda^*_N(\mu^{c,\alpha}_{N,a}) \ = \ \frac{1}{Z^{c,\alpha}_{N,a}} \, e^{-\langle U_0(k), 1\rangle} \, \Lambda^*_N(\mu_{N,a})(dk).$$

We denote by K_w the set K endowed with the weak topology of $L^2(0, 1)$ and by $C_b(K_w) \subset C_b(K)$ the space of functions on K, bounded and uniformly continuous w.r.t. the weak topology.

Theorem 3. *Let* $\alpha, c > 0$.

1. *For all* $T > 0$, $\gamma > 1$, $(\Phi_{N,0}^{c,\alpha}(t))_{t\in[0,T]}$ *converges in law to* $(u_0^{c,\alpha}(t))_{t\in[0,T]}$ *in* $C([0,T]; H^{-\gamma}(0,1))$ *as* $N \to \infty$.
2. *Let* γ_N *be any* K_N*-valued r.v. independent of* $\Phi_{N,0}^{c,\alpha}$ *and* γ *any* K*-valued r.v. independent of* $u_0^{c,\alpha}$, *such that* $\gamma_N \to \gamma$ *in distribution in* $L^2(0,1)$. *Then for all* $f \in C_b(K_w^n)$, $0 < t_1 < \cdots < t_n$*:*

$$\mathbb{E}\left[f(\Phi_{N,0}^{c,\alpha}(t_i, \gamma_N), i = 1, \ldots, n)\right] \overset{N\to\infty}{\longrightarrow} \mathbb{E}\left[f(u_0^{c,\alpha}(t_i, \gamma), i = 1, \ldots, n)\right].$$

Proof. We apply Theorem 2 to the reflected interface $\Phi_N^{U_\epsilon}$ and, using the monotonicity arguments of [11] and [17], we obtain the thesis letting $\epsilon \to 0$.

Let $\mathcal{H}$ be $L^2(0,1)$ or $L^2((0,T)\times(0,1))$. Given two probability measures μ and ν on $\mathcal{H}$, we say that $\mu \geq \nu$ if $\int_K F\,d\mu \geq \int_K F\,d\nu$ for all Borel bounded $F : \mathcal{H} \mapsto \mathbb{R}$ such that $k \geq k'$ in $\mathcal{H}$ implies $F(k) \geq F(k')$. We divide the proof into several steps.

Step 1. We prove first that $m_{N,0}^{c,\alpha}$ converges to $m_0^{c,\alpha}$ in K_w. We follow the proof of Lemma 3.1 of [8]. We first recall that tightness in K_w of a sequence of probability measures $(M_n)_n$ is equivalent to: $\lim_{L\to\infty}\sup_n M_n(k : \|k\| \geq L) = 0$. If $a > 0$ for m^a-a.e. ω we have $\omega > 0$ over $[0,1]$. Then by (41) it is easy to prove that $m_{N,a}^{c,\alpha}$ converges to $m_a^{c,\alpha}$ as $N \to \infty$ in $D([0,1])$. Let

$$\Psi_t := \|(\Phi_{N,0}^{c,\alpha}(t,k) - \Phi_{N,a}^{c,\alpha}(t,k))^+\|^2 + \|(\Phi_N^{U_\epsilon}(t,k) - \Phi_{N,0}^{c,\alpha}(t,k))^+\|^2,$$

where $(r)^+ := r \vee 0$ and $t \geq 0, k \in K_N$. Applying the Itô formula to Ψ and arguing as in the proof of (7), we obtain $d\Psi_t/dt \leq 0$ and therefore:

$$\Phi_{N,a}^{c,\alpha}(t,k) \geq \Phi_{N,0}^{c,\alpha}(t,k) \geq \Phi_N^{U_\epsilon}(t,k), \qquad t \geq 0,\ k \in K_N. \tag{42}$$

By the ergodicity this implies $m_{N,a}^{c,\alpha} \geq m_{N,0}^{c,\alpha} \geq m_N^{U_\epsilon}$. In particular we obtain tightness of $(m_{N,0}^{c,\alpha})_N$ in K_w. Let $\hat{m}$ be any limit point of $(m_{N,0}^{c,\alpha})_N$. By the above considerations: $m_a^{c,\alpha} \geq \hat{m} \geq m^{U_\epsilon}$, $a, \epsilon > 0$. By point 3 of Proposition 2, $m^{U_\epsilon} \to m_0^{c,\alpha}$ as $\epsilon \to 0$ and $m_a^{c,\alpha} \to m_0^{c,\alpha}$ as $a \to 0$, so that $\hat{m} = m_0^{c,\alpha}$.

Step 2. We prove now point 1, following the proof of Theorem 1.1 in [8]. Tightness of $(\Phi_{N,0}^{c,\alpha}(t))_{t\in[0,T]}$ follows from Lemma 7, since the estimate in (36) is uniform in $(U_\epsilon)_{\epsilon>0}$, and from the tightness of $(m_{N,0}^{c,\alpha})_N$ in K_w.

By the second inequality in (42), we have that $P_N^{c,\alpha} \geq P_N^{U_\epsilon}$, where $P_N^{c,\alpha}$ and $P_N^{U_\epsilon}$ denote, respectively, the law of $\Phi_{N,0}^{c,\alpha}$ and $\Phi_N^{U_\epsilon}$. If P is any weak limit of $(P_N^{c,\alpha})_N$, then by point 1 of Theorem 2: $P \geq P^{U_\epsilon} :=$ law of v^{U_ϵ}. By the results of [17], as $\epsilon \to 0$ we have $v^{U_\epsilon} \to u_0^{c,\alpha}$. Therefore we obtain the *dynamical lower bound*: $P \geq P^{c,\alpha} :=$ law of $u_0^{c,\alpha}$.

On the other hand, since the law of $\Phi_{N,0}^{c,\alpha}(t)$ is $m_{N,0}^{c,\alpha}$ for all N, then we have by the previous step the *static equality*: $P(\Phi(t) \in \cdot) = m^{c,\alpha} = P^{c,\alpha}(\Phi(t) \in \cdot)$, where Φ is the coordinate process on $C([0,\infty)\times H^{-\gamma}(0,1))$. The dynamical lower bound and the static equality yield the result.

Step 3. We prove now point 2. By (37) it is enough to consider deterministic $\gamma \equiv k \in K$ and $\gamma_N \equiv \Pi_N k$. By point 1, $((\Phi^{c,\alpha}_{N,0}(t_i), i = 1, \ldots, n)_N$ is tight in K^n_w, so that, by (37), $((\Phi^{c,\alpha}_{N,0}(t_i, k_N), i = 1, \ldots, n)_N$ is tight in K^n_w. Notice that for all $h \in K$, the map $K \ni k \mapsto \exp(-\langle h, k\rangle) =: \varphi_h(k)$ is monotone non-increasing. Set for $k \in K$:

$$F_N(k) := \mathbb{E}[\varphi_h(\Phi^{c,\alpha}_{N,0}(t, \Pi_N k))], \qquad F_{N,\epsilon}(k) := \mathbb{E}[\varphi_h(\Phi^{U_\epsilon}_N(t, \Pi_N k))].$$

By (25), arguing like in the proof of Lemma 6 from every subsequence of $(F_N)_N$ we can extract a sub-subsequence, which we still denote by $(F_N)_N$, converging pointwise to a function $F \in C_b(K)$. Since $k \mapsto \Phi^{c,\alpha}_{N,0}(t, k)$ is a monotone non-decreasing map, then F_N is monotone non-increasing. By (42) and the weak convergence in H of $(m^{c,\alpha}_{N,a})_N$ and $(m^{U_\epsilon}_N)_N$:

$$m^{c,\alpha}_a(F) = \lim_N m^{c,\alpha}_{N,a}(F_N) \le \liminf_N m^{c,\alpha}_{N,0}(F_N)$$

$$\le \limsup_N m^{c,\alpha}_{N,0}(F_N) \le \lim_N m^{U_\epsilon}_N(F_N) = m^{U_\epsilon}(F)$$

and letting $a, \epsilon \to 0$ we obtain $m^{c,\alpha}_{N,0}(F_N) \stackrel{N\to\infty}{\longrightarrow} m^{c,\alpha}_0(F)$ by point 3 of Proposition 2. Therefore

$$\int F\, dm^{c,\alpha}_0 = \lim_N \int F_N\, dm^{c,\alpha}_{N,0} = \lim_N \int \varphi_h\, dm^{c,\alpha}_{N,0} = \int \varphi_h\, dm^{c,\alpha}_0$$

$$= \int \mathbb{E}\left[\varphi_h(v^{c,\alpha}_0(t, k))\right] m^{c,\alpha}_0(dk). \tag{43}$$

Now, by point 2 in Theorem 2 and (42):

$$F(k) \stackrel{N\to\infty}{\longleftarrow} F_N(k_N) \le F_{N,\epsilon}(k_N) \stackrel{N\to\infty}{\longrightarrow} \mathbb{E}[\varphi_h(v^{U_\epsilon}(t, k))],$$

so that letting $\epsilon \to 0$: $F(k) \le \mathbb{E}[\varphi_h(v^{c,\alpha}_0(t, k))]$. Therefore by (43): $F(k) = \mathbb{E}[\varphi_h(v^{c,\alpha}_0(t, k))]$ for $m^{c,\alpha}_0$-a.e. k and, by continuity, for all $k \in K$. This allows by (25), the Markov property and the uniqueness of the Laplace transform in K^n_w, to obtain the thesis. □

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