

# Zero modes in de Sitter background

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**ABSTRACT:** There are five well-known zero modes among the fluctuations of the metric of de Sitter (dS) spacetime. For Euclidean signature, they can be associated with certain spherical harmonics on the  $S^4$  sphere, viz., the vector representation  $\mathbf{5}$  of the global  $SO(5)$  isometry. They appear, for example, in the perturbative calculation of the on-shell effective action of dS space, as well as in models containing matter fields. These modes are shown to be associated with collective modes of  $S^4$  corresponding to certain coherent fluctuations. When dS space is embedded in flat five dimensions  $E^5$ , they may be seen as a legacy of translation of the center of the  $S^4$  sphere. Rigid translations of the  $S^4$ -sphere on  $E^5$  leave the classical action invariant but are unobservable displacements from the point of view of gravitational dynamics on  $S^4$ . Thus, unlike similar moduli, the center of the sphere is not promoted to a dynamical degree of freedom. As a result, these zero modes do not signify the possibility of physically realizable fluctuations or flat directions for the metric of dS space. They are not associated with Killing vectors on  $S^4$  but can be identified with certain non-isometric, conformal Killing forms that locally correspond to a rescaling of the volume element  $dV_4$ .

We frame much of our discussion in the context of renormalizable gravity, but, to the extent that they only depend upon the global symmetry of the background, the conclusions should apply equally to the corresponding zero modes found in Einstein gravity. Although their existence has only been demonstrated at one-loop, we expect that these zero modes will be present to all orders in perturbation theory. They will occur for Lorentzian signature as well, so long as the hyperboloid  $H^4$  is locally stable, but there remain certain infrared issues that need to be clarified. We conjecture that they will appear in any gravitational theory having dS background as a locally stable solution of the effective action, regardless of whether additional matter is included.

**KEYWORDS:** Space-Time Symmetries, Differential and Algebraic Geometry, Models of Quantum Gravity

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**1 Introduction**

There are five well-known zero modes in the conformal fluctuations of the metric of de Sitter (dS) space. For Euclidean signature, they are associated with the spherical harmonics on the sphere  $S^4$  corresponding to the vector representation  $\mathbf{5}$  of  $SO(5)$ . These five zero modes are ubiquitous, appearing in renormalizable gravity, both with and without additional matter, as well as in loop corrections to the usual Einstein-Hilbert (E-H) theory, treated as an effective field theory.  $S^4$  may be embedded in flat five-dimensional spacetime  $E^5$ , whose isometries are the Poincaré group,  $SO(5) \times P^5$ . We shall show that the invariance of the embedding under translations in five-dimensions ( $P^5$ ) is reflected by certain collective modes or moduli that leave the gravitational action in four dimensions invariant. In the transverse-traceless gauge, these can be associated with certain conformal fluctuations of the metric on  $S^4$ .

For Lorentzian signature, a similar analysis is expected to apply to the hyperboloid  $H^4$  with isometry  $SO(4,1)$ , although there are subtleties that have not been resolved stemming from the long-range behavior of the fluctuations. We do not believe this infrared issue represents an insuperable obstacle to analytic continuation from Euclidean to Lorentzian signature.

These zero modes appear to be a universal feature of models in dS space, for reasons that will be explained in this paper. Our point of view regarding Euclidean quantum gravity is more or less the same as that expressed by Christensen & Duff [1], except that we extend that philosophy to renormalizable gravity. The E-H theory has well known instabilities in the conformal sector, and it has been suggested [2] that the contour of integration in the Euclidean path integral (EPI) be changed for these unstable modes. Even if one adopts their prescription, these five zero modes persist. However, some may take the point of view that the entire framework is suspect as a result of those instabilities. One advantage of renormalizable gravity is that, with a sensible choice of the sign of the coupling constants, there is no need to modify the definition of the EPI to achieve convergence in the conformal sector. Further, for a subset of this range of couplings, there are no unstable modes for fluctuations about dS background at one-loop order [3, 4]. Nevertheless, there remain the five zero modes that are the focus of this paper. A second advantage of renormalizable gravity is that it is asymptotically free in the gravitational couplings [3–8]. In certain circumstances, asymptotic freedom may be extended to all couplings<sup>1</sup> [10–13]. As a result, perturbation theory can be trusted at sufficiently high scales.

As did the authors of ref. [1], we reject the notion that spacetime is asymptotically flat, since that is not a solution of the field equations in the presence of a nonzero cosmological constant. Correspondingly, we cannot assume the existence of an S-matrix but instead emphasize correlation functions through the perturbative calculation of the effective action  $\Gamma[g_{\alpha\beta}]$ . (For the same reason, it is also important not to discard the Gauss-Bonnet term [1, 14], which is nonzero at every point.)

Some of our results overlap with a paper by Gibbons & Perry<sup>2</sup> [15]. In particular, in their section 2, they cite a theorem [16] that, assuming Euclidean signature, these non-isometric, conformal zero modes can only occur in  $d = 4$  for  $S^4$ . (Surprisingly, these modes are passed over in their treatment of Euclidean dS space in their section 4.) The extension of this theorem to the pseudo-Riemannian case has been addressed subsequently. Assuming Einstein’s equations in vacuum, a much stronger assumption than assuming that spacetime is an Einstein space, the theorem can be extended to Lorentzian signature [17], implying the spacetime is either dS or anti-de Sitter (AdS). This topic has also received further attention in the mathematical literature; for a recent review and discussion, see ref. [18]. Under various technical assumptions, much weaker than requiring Einstein’s equations, such non-isometric, conformal zero modes for Lorentzian signature can only occur in a spacetime of constant curvature. In general, the manifold need not be simply connected, so it cannot be inferred that the spacetime is dS or AdS without additional assumptions.

Folacci [19, 20] has also discussed the nature of these five zero modes, suggesting that they should be regarded as gauge artifacts. In ref. [1], it is shown that one obtains agreement between the general result in an arbitrary background and the dS result only if these zero modes are counted. Since these modes are present on-shell, it seems doubtful that they are true gauge artifacts, but their unphysical nature does resemble a gauge symmetry. For

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<sup>1</sup>The work by Buchbinder and collaborators is reviewed in detail in chapter 9 of ref. [9].

<sup>2</sup>Ref. [1] corrects some numerical errors in ref. [15], but our results do not depend upon such details.

Lorentzian signature, although unproven, our point of view is also different from his, an issue to which we shall return in section 6.

In the next section, we review some aspects of the background field method for calculating the effective action in perturbation theory. Then, in section 3, we describe the embedding of Euclidean dS space as a submanifold in flat five dimensions. In section 4, we do the reverse, explaining how one may lift metric fluctuations from four to five dimensions. Finally, in section 5, we interpret the five zero modes as remnants of a potential collective mode in five dimensions. Some comments concerning the extension to Lorentzian signature are contained in section 6. Finally, further discussion and conclusions follow in sections 7 and 8, respectively.

## 2 Effective action

The calculation of the effective action<sup>3</sup> of a quantum field theory (QFT) is one of the most useful ways to explore its properties. The perturbative calculation of the effective action in quantum gravity has a long history.<sup>4</sup> In general, it is technically complicated by the large gauge symmetry (diffeomorphism invariance) and tensorial calculus, as well as by the conceptual issues associated with the fact that, in a sense, the spacetime itself is determined self-consistently by the calculation.

Our interest was stimulated in part by Avramidi’s calculation [3, 4] of the effective potential for the curvature of dS space in renormalizable gravity, but our result concerning these five zero modes depends only upon the symmetries of the background field. Conceptually, it is somewhat simpler to begin with a renormalizable theory in which the EPI is well-defined. Like Avramidi, we can assume that the action has both an E-H term as well as a cosmological constant, assumed positive. (There are slight but important changes required to accommodate the classically scale invariant cases of interest in our other recent work [13, 21, 22].) We shall deal here with the purely gravitational case, but it will be self-evident that it can be generalized to the inclusion of matter fields in dS background.

The calculation of the effective action

$$\Gamma_{\text{eff}}[g_{\alpha\beta}(x)] \equiv S_{cl}[g_{\alpha\beta}(x)] + \Delta\Gamma_{\text{eff}}[g_{\alpha\beta}(x)] \tag{2.1}$$

in perturbation theory is somewhat simplified by the background field method, which is most easily described in terms of the EPI. To establish notation, the classical action will be expressed as

$$S_{cl} = \int d^4x \sqrt{g} \left[ \frac{C^2}{2a} + \frac{R^2}{3b} + cG - \frac{M_P^2}{2}(R - 2\Lambda) \right], \tag{2.2}$$

where  $C \equiv C^\gamma_{\epsilon\alpha\beta}$  is the Weyl tensor;  $G$ , the Gauss-Bonnet term  $G \equiv C^2 - 2W$ ;  $W \equiv R^2_{\alpha\beta} - R^2/3$ ;  $R_{\alpha\beta}$ , the Ricci tensor; and  $R$ , the scalar curvature.  $M_P$  is the reduced Planck mass or string scale, and  $\Lambda$  is the cosmological constant, both assumed positive. The

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<sup>3</sup>There are numerous effective actions that have been defined. In this paper, we shall only employ the generating functional  $\Gamma$  of 1PI Green’s functions, the Legendre transform of  $W[J]$ . It is gauge-dependent although its value at an extremum is not.

<sup>4</sup>See refs. [3, 4, 9] for extensive reviews.

maximally symmetric solution of the classical field equations has the same form as in E-H gravity, viz.,  $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$ , with a metric  $g_{\alpha\beta}^{dS}(x)$  that describes dS space. (Its application to cosmology requires certain additional assumptions that will not be taken up here.)

We shall next summarize the use of the background field method to calculate the effective action perturbatively.<sup>5</sup> Our purpose for reviewing this is to clarify what the fields  $h_{\alpha\beta}$  represent from the point of view of the EPI, which, because the theory is required to be invariant under arbitrary diffeomorphic transformations of the metric, is most clearly expressed in the language of differential geometry. (Readers familiar with the effective action and with the background field method may skip to the next section.) To proceed, one places this classical action in the EPI and attempts to integrate over all metrics  $g_{\alpha\beta}(x)$  under certain consistency conditions. Convergence of the EPI requires that the couplings  $a, b > 0$  (with the G-B coupling  $c$  determined by  $a, b$  up to an additive constant [14]). To perform the EPI, the metric is split

$$g_{\alpha\beta}(x) \equiv g_{\alpha\beta}^B(x) + h_{\alpha\beta}(x), \tag{2.3}$$

where the “classical” background field  $g_{\alpha\beta}^B(x)$  is generically an arbitrary function to be determined, and the quantum field  $h_{\alpha\beta}(x)$  is to be integrated out.  $h_{\alpha\beta}(x)$  will be referred to as the quantum fluctuations or simply fluctuations. The Feynman rules for  $h_{\alpha\beta}(x)$  are obtained in principle by expanding  $S[g_{\alpha\beta}^B(x) + h_{\alpha\beta}(x)]$  in powers of  $h_{\alpha\beta}(x)$  and dropping the linear term. The consistency condition mentioned above is that the one-point function for  $h_{\alpha\beta}$  vanish to all orders, i.e., the classical field is in fact the background field,  $\langle g_{\alpha\beta} \rangle = g_{\alpha\beta}^B$ . Said otherwise, the effective action is the generating functional of one-propagator-irreducible<sup>6</sup> (1PI) Feynman diagrams in the presence of a classical background  $g_{\alpha\beta}^B(x)$ . Its extrema,

$$\frac{\delta\Gamma_{\text{eff}}[g_{\alpha\beta}^B]}{\delta g_{\alpha\beta}^B(x)} = 0, \tag{2.4}$$

replace the classical equations of motion (EOM). When eq. (2.4) is satisfied, the background field is said to be “on-shell.” Stability of the solution is investigated by evaluating higher-order variations on-shell. In principle, this should be carried out for an arbitrary background field but, in practice, it is often restricted by certain assumptions about the relevant global isometries.

This seemingly circular procedure for determining  $g_{\alpha\beta}^B$  is, in principle, straightforward to implement in perturbation theory, although in practice, it can seldom be carried out explicitly without further approximations. In lowest order, the background field is approximated by the solution of the (renormalized) classical field equations  $g_{\alpha\beta}^B(x) \rightarrow g_{\alpha\beta}^{cl}(x)$ , which may receive quantum corrections in higher order. Following the procedure described above, the propagator and vertices depend explicitly on the background field. The one-loop result, which is as far as these calculations have been carried in renormalizable quantum gravity, is

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<sup>5</sup>In the present context, a brief overview can be found in an appendix to ref. [21].

<sup>6</sup>Given that the notion of “particle” is frame dependent and the propagator refers to the quanta associated with fluctuations  $h_{\alpha\beta}$  in a certain background field, this appellation seems more appropriate than the usual nomenclature “one-particle-irreducible”.

determined by the quadratic terms alone, which take the form of a functional integral over  $\exp[-h_{\gamma\epsilon}(x)\mathcal{O}^{\gamma\epsilon\alpha\beta}h_{\alpha\beta}(x)]$ , for some local differential operator  $\mathcal{O}^{\gamma\epsilon\alpha\beta}(g_{\rho\sigma}^B(x), \nabla_\tau^B)$ , where  $\nabla^B$  represents the covariant derivative associated with the background metric.

Although the fluctuations  $h_{\alpha\beta}(x)$  have no particular symmetry, in the case at hand, the operators  $\mathcal{O}^{\gamma\epsilon\alpha\beta}(x)$  will be restricted by the SO(5) global symmetry presumed of the background. In order to carry out the integration, it must be that the eigenvalues of  $\mathcal{O}^{\gamma\epsilon\alpha\beta}(g_{\rho\sigma}^B(x), \nabla_\tau^B)$ , are non-negative, at least in a neighborhood of being on-shell. Even if there are no truly unstable fluctuations, there may occur certain “flat directions” or “zero modes”, i.e., field configurations  $h_{\alpha\beta}^{(z)}(x)$  that make no change in the value of the action. To one-loop order, these can be expressed by the partial differential equation

$$\mathcal{O}^{\gamma\epsilon\alpha\beta}(g_{\kappa\lambda}^{cl}(x); \nabla_\tau^{cl})h_{\alpha\beta}^{(z)}(x)=0, \tag{2.5}$$

in which the background field is taken to be a solution to the classical EOM.

The occurrence of zero modes in QFT is seldom accidental and usually reflects some symmetry of the theory, either unbroken or spontaneously broken, or the identification or emergence of some collective coordinate. In the case of spontaneous broken gauge theories, massless Goldstone bosons persist unless absorbed by giving mass to some vector bosons. Sometimes, pseudo-Goldstone bosons occur because of some symmetry of the dimension-four scalar interactions that is not a symmetry of the full QFT [23]. In principle, one may try to calculate two-loop and higher corrections to determine whether these zero modes persist, but, in these gravitational models, it is usually prohibitively complicated to carry out.

Local gauge symmetries complicate the issue further because they guarantee that certain transformations of a vector field or metric are physically equivalent and have no effect on-shell. This implies that the fluctuations may be subdivided into equivalence classes wherein each element is a gauge-transform of another. Aside from lattice gauge theory, the only way found so far to deal with this redundancy is to select a single representative or a subset of representatives of each equivalence class by means of “gauge-fixing” constraints that allow the propagator to be determined and to add so-called Faddeev-Popov ghost fields to ensure that the final result is independent of the representative chosen.<sup>7</sup> As if this were not confusing enough, in gravity, the very choice of a coordinate system in which to express the background metric  $g_{\alpha\beta}^B(x)$  already involves at least a partial choice of gauge. Unfortunately, for curved spacetime, no coordinate-independent method of calculation has been found. To calculate to higher-order, after gauge-fixing, the quadratic form involving a modified operator  $\tilde{\mathcal{O}}^{\gamma\epsilon\alpha\beta}(\nabla^B; g_{\kappa\lambda}^B)$  is inverted to define propagators for  $h_{\alpha\beta}(x)$ , and the terms cubic and higher-order in  $h_{\alpha\beta}$  determine the “interaction vertices”. Although the initial form of the second-order terms does not require gauge-fixing or even a specific choice of coordinates, the actual evaluation of the functional determinants arising at one-loop and the calculations at two-loops and higher do. Consequently, in general, the result for  $\Gamma_{\text{eff}}$  will depend on the choice of gauge; however, “on-shell,” when eq. (2.4) is

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<sup>7</sup>In fact, this technique has evolved into “gauge-averaging” rather than gauge-fixing. We have not seen this treated in the mathematical physics literature.

satisfied, this gauge dependence must disappear. As summarized above, to one-loop order, the first approximation to these on-shell conditions correspond simply to solutions of the classical EOM.

Even so, as in ordinary QFT in Minkowski space, the one-loop effective action cannot be evaluated analytically (or numerically) except in certain very special backgrounds. In non-gravitational models, in the case of spacetime independent background fields, the effective action reduces to an effective potential, whose generic form is known. The most nearly analogous case in gravity corresponds to a maximally-symmetric background metric, such as dS or AdS, together with constant matter fields, if present. Unfortunately, the generic form of the one-loop potential is not known in this case. Nevertheless, the one-loop calculation in dS for pure higher-derivative gravity has been carried out in certain cases, and the beta-functions, which *are* gauge independent, have been determined in general [3, 5–8]. In particular, Avramidi [3, 4] showed that, with certain restrictions on the range of coupling constants, the second-order fluctuations in dS background were all stable on-shell with the exception of the five zero modes in the conformal sector. Although he believed them to be accidental and destabilizing beyond one-loop, the arguments in this paper suggest that they are a consequence of dS background and will persist to all orders in perturbation theory, at least for on-shell quantities.

In Euclidean quantum gravity, the dS background is regarded as the sphere<sup>8</sup>  $S^4$ . Assuming the background is SO(5) invariant, it is useful to expand the field  $h_{\alpha\beta}$  in representations of SO(5) because it diagonalizes the operators  $\tilde{\mathcal{O}}^{\gamma\epsilon\alpha\beta}$  (for a judicious choice of gauge-fixing). Of course, since the full isometry is not manifested by any choice of coordinates, normally an investigation of the Killing equations must be carried out:

$$\nabla_a \xi_b + \nabla_b \xi_a = \nabla \cdot \xi \frac{g_{ab}}{2}. \tag{2.6}$$

If  $\xi^a$  is closed, i.e.,  $\nabla \cdot \xi = 0$ , the Killing vector field  $\xi^a(x)$  is a generator of an isometry of the metric. (On a contractible manifold, such as  $S^4$ , a closed vector is exact  $\xi_a = \nabla_a X$  for some function  $X$ .) If  $\nabla \cdot \xi \neq 0$ , then  $\xi^a$  is a non-isometric, conformal Killing field or a homothetic field. As the name suggests, it is associated with a conformal transformation of the metric.

A more intuitive approach to dS isometries is to embed  $S^4$  into flat Euclidean spacetime in five dimensions,  $E^5$ , where the metric,  $\delta_{ij}$ , is trivial in Cartesian coordinates, and the SO(5) isometry is manifest. The expansion of the fluctuations in irreducible representations (irreps) of SO(5) is far more easily performed in terms of tensors on the co-tangent bundle on  $E^5$  rather than on  $S^4$ . In the next section, we shall review classical dS space as a submanifold in five-dimensional flat space.

### 3 De Sitter space as a submanifold in five dimensions

As mentioned earlier, for the time being, we shall work with Euclidean signature and treat classical dS space as the sphere  $S^4$ . Later, in section 6, we shall comment on what changes

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<sup>8</sup>If one assumes less than maximal symmetry, other topologies can also be entertained, but, as discussed in section 1, non-isometric conformal solutions do not exist except for  $S^4$ .

are required for Lorentzian signature. We begin by describing  $S^4$  as an embedding in flat, five dimensional space  $E^5$  which, in Cartesian coordinates has the trivial metric:

$$ds^2 = \delta_{ij} dx^i dx^j. \tag{3.1}$$

$S^4$  may be defined as the set of all points  $x^i$  in  $E^5$  satisfying the equation

$$\delta_{ij} x^i x^j \equiv \|x\|^2 = r_0^2, \tag{3.2}$$

where  $r_0$  is related to the on-shell value  $R_0 \equiv 4\Lambda$  of the scalar curvature by  $r_0 \equiv \sqrt{12/R_0}$ . It must be shown that this embedding actually corresponds to the dS metric on  $S^4$ , but we shall take that as given. By inclusion,<sup>9</sup> every point on  $S^4$  can be assigned coordinates in  $E^5$ , so one has a mapping from  $S^4$  to  $E^5$ . The co-tangent space on  $S^4$  may be regarded as the pull-back of the co-tangent space on  $E^5$ .

The beauty of this description is that it does not require the explicit introduction of coordinates on  $S^4$ , so that the isometries of dS space are transparent. That is useful because no single coordinate system covers all of  $S^4$ , and the selection of any particular coordinate system only reveals a subset of the isometries of the  $S^4$  submanifold. For example, much of  $S^4$  is covered by spherical coordinates, delineated explicitly in eq. (A.2b), but the only evident isometry is the independence of the metric on the angle  $\theta_4$ . In fact,  $S^4$  is invariant under global  $SO(5)$ , as is easily seen, since the submanifold of  $E^5$  is completely specified by eq. (3.2).

The embedding of  $S^4$  in  $E^5$  is a purely geometrical construction in order to clarify the isometries of the background field. It has no effect on the dynamics, which always takes place in four-dimensions. Nevertheless, we can describe the decomposition of the fluctuations into irreps of  $SO(5)$  more clearly in this manner. To understand why, we need to spell out the nature of the calculation of the Euclidean path integral (EPI). This is a rather long detour into explaining the decomposition of fluctuations of the metric in dS space into harmonics of  $SO(5)$ . As a bonus, however, we shall finally understand the origin of the five zero modes.

The construction to follow expands somewhat on discussions given previously in refs. [3, 15, 27]. Although  $h_{\alpha\beta}(x)$  is a fluctuation of the metric, the point  $x$  always remains on the background manifold. At every point  $x$  on the  $S^4$  manifold, the tangent space  $TS^4_x$  is defined and a set of basis vectors chosen. It can be either a coordinate basis  $\partial_\alpha$  or a linearly independent tetrad or vierbein  $e_a(x) = e_a^\alpha(x)\partial_\alpha$ , conventionally taken to be orthonormal in the sense that

$$e_a^\alpha(x)e_b^\beta(x)g_{\alpha\beta}^B(x) = \delta_{ab}, \quad \text{and, inversely, } e_\alpha^a(x)e_\beta^b(x)\delta_{ab} = g_{\alpha\beta}^B(x), \tag{3.3}$$

where  $g_{\alpha\beta}^B(x)$  is the metric on the background  $S^4$ . The components of a tangent vector  $v$  can be defined either by  $v = v^\alpha\partial_\alpha$  or, alternatively, by  $v = v^a e_a$ , with the relationship between the two descriptions given by the invertible matrix  $e_a^\alpha(x)$ , i.e.,  $v^\alpha = v^a e_a^\alpha$ . Similarly,  $dx^\alpha = e_a^\alpha(x)dx^a$ , so that an arbitrary one-form  $dv$  may be expanded either way, with

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<sup>9</sup>Our notation and some basic concepts in differential geometry are summarized in appendix A.

coordinates related by  $dv_\alpha = dv_a e_\alpha^a(x)$ . The collection of all such vectors  $\mathbf{v}$  is called the tangent fiber at the point  $x$ , which is a four-dimensional vector space  $TS_x^4$ . Each point  $x$  is associated with a different tangent space, so the elements of each tangent space requires pairing the point  $x$  together with the coordinates of vectors of the fiber. The collection of all such points with their associated fibers, for all possible choice of coordinates, forms the tangent bundle  $TS^4$ . It is eight-dimensional, requiring four coordinates to specify the point, and another four to label the coordinates of each tangent vector.

Similarly, the co-tangent bundle  $T^*S^4$  has fibers consisting of the cotangents  $dv$  expressed in either form. The background metric  $g_{\alpha\beta}^{dS}(x)$  on  $S^4$  and the fluctuations  $h_{\alpha\beta}(x)$  are the components of symmetric, bi-linear functionals defined on the co-tangent space at  $x$ , e.g., in a coordinate basis,  $h_{\alpha\beta}(x)dx^\alpha dx^\beta$ . The EPI integrates over all the fluctuations  $h_{\alpha\beta}(x)$  at each point as well as over all points on  $S^4$ .

One may perform a similar construction on  $E^5$ , forming the tangent  $TE^5$  and co-tangent  $T^*E^5$  bundles. Since it is five-dimensional, we must introduce a fünfbein basis  $e_a(x) \equiv e_a^i(x)\partial_i, \{a = 1, \dots, 5\}$ , which is trivial in Cartesian coordinates  $e_a^i \equiv \delta_a^i$ . The essential difference between the vierbein basis  $e_a(x)$  of  $TS^4$  and the fünfbein basis for  $E^5$  is of course the radial vector, represented in Cartesian coordinates by  $e_r^i(x) \equiv x^i/r \equiv \hat{x}^i$ , where  $r = \|x\|$ . Unlike eq. (3.2), here  $r$  need not be confined to the original  $S^4$  submanifold with  $r = r_0$ . (It may be useful to keep in mind spherical coordinates, explicitly given in eq. (A.1), for which the metric takes the form of eq. (A.2).)

When we say that the metric on  $S^4$  is the pull-back of the metric on  $E^5$ , what we mean is that, since the co-tangent bundle  $T^*S^4$  is the pull-back of  $T^*E^5$ , covariant tensors of the latter, such as the metric or  $p$ -forms, may be associated with covariant tensors of the former. Since  $h_{\alpha\beta}(x)$  is arbitrary, the co-tangent bundle  $T^*E^5$  will not be  $SO(5)$ -invariant, but this pull-back, a linear transformation, can be carried out regardless because the arguments of these tensor fields lie on  $E^5$  and  $S^4$ . For example, in spherical coordinates, the  $E^5$  metric takes the form given in eq. (A.2). To pull back to the  $S^4$  manifold, one may simply fix  $r = r_0$  in eq. (A.2a) and drop the  $dr$  term to get  $ds_4^2 = r_0^2 d\omega_4^2$ . The dimensionless quantity  $d\omega_4^2$ , eq. (A.2b), denotes the metric on the unit  $S^4$  sphere in these coordinates.

Since one can always associate an external normal at a point on an orientable manifold, it is worth asking whether some geometrical feature of fluctuations on  $S^4$  can be associated with the radial direction in  $E^5$ . A hint can be found in the Hodge-dual or Hodge-star of the one-form  $e^r(x) \equiv e_i^r(x)dx^i$ :

$$*e^r(x) = e^{a_1}(x) \wedge e^{a_2}(x) \wedge e^{a_3}(x) \wedge e^{a_4}(x), \quad (3.4)$$

where the  $e^{a_j}(x)$  are an (appropriately ordered) orthonormal basis of  $T^*S^4$  at the point  $x$ . The right-hand side of eq. (3.4) is proportional to the coordinate-invariant volume co-form on  $S^4$ :

$$dV_4 \equiv \sqrt{g_4} e^{a_1}(x) \wedge e^{a_2}(x) \wedge e^{a_3}(x) \wedge e^{a_4}(x). \quad (3.5)$$

Thus, up to the factor of  $\sqrt{g_4}$ , we may identify the radial direction in  $E^5$  with  $dV_4$ .

Another way to view this relation is to start from the volume co-form in  $E^5$ :

$$dV_5 \equiv \sqrt{g_5} e^r(x) \wedge e^{a_1}(x) \wedge e^{a_2}(x) \wedge e^{a_3}(x) \wedge e^{a_4}(x). \quad (3.6)$$

The relations above can be conveniently stated in a coordinate independent fashion in terms of the interior product  $\iota_v$ , which maps forms of order  $p$  to forms of order  $p-1$ , such as the mapping from  $dV_5$  to  $dV_4$ . (See appendix A.) In this language, the unit area of  $S^4$  at a point  $x$  is given by the four-form

$$dV_4 = \iota_{e_r}[dV_5]. \tag{3.7}$$

Correspondingly, a radial vector  $v(x)e_r(x)$  at a point  $x$  is dual to a rescaled local volume element  $v(x)dV_4$  on  $S^4$ . (N.B.  $g_4 \neq g_5|_{r=r_0}$ .)

The upshot of this is that the differential “surface area”  $dV_4$  on  $S^4$  may be associated with the contraction of the five-form volume  $dV_5$  with the unit normal  $\hat{e}_r(x)$  in  $TE^5$ , evaluated on the submanifold  $r = r_0$ . If the  $S^4$  metric fluctuates in a way that changes its surface area (volume form on  $S^4$ ) by some amount, it can equivalently be expressed as a certain *rescaling* of the magnitude of the normal vector in  $E^5$ . This relationship is a key to understanding how conformal fluctuations of the volume form on  $T^*S^4$  are related to radial fluctuations on  $T^*E^5$ .

#### 4 Lifting metric fluctuations from $S^4$ to $E^5$ .

Now we wish to consider metric fluctuations  $h_{\alpha\beta}$  on  $T^*S^4$  and, in particular, to spell out how they are reflected in  $T^*E^5$ . To simplify the discussion, we shall suppress gauge-dependent fluctuations and work with those that survive on-shell. For this purpose, it is helpful to choose a unitary gauge in which the gauge degrees of freedom vanish, such as the transverse-traceless (TT) gauge in which

$$h_{\alpha\beta} = \frac{h}{4}g_{\alpha\beta}^B + h_{\alpha\beta}^\perp, \tag{4.1}$$

where  $h \equiv g^{B\alpha\beta}h_{\alpha\beta}$  and  $\nabla^{B\alpha}h_{\alpha\beta}^\perp = 0$ , setting the other four gauge-dependent modes to zero.<sup>10</sup> The fluctuations  $h_{\alpha\beta}^\perp(x)$  form a symmetric traceless tensor on the co-tangent space  $T^*S_x^4$  that vanish when evaluated on the normal  $e_r(x)$ , which can easily be visualized from the embedding of  $TS^4$  in  $TE^5$ . Thus, they do not change the volume element  $dV_4$  at  $x$ . On the other hand, the conformal fluctuations

$$g_{\alpha\beta}^B(x) \left(1 + \frac{h(x)}{4}\right) \tag{4.2}$$

change the volume form  $dV_4$  through its rescaling of  $\sqrt{g_4(x)}$  by  $(1+h(x)/4)^2$ . As discussed in the preceding section, via the Hodge dual, this can be pictured as a change in the radial component  $g_{rr}dr^2$  of the metric on  $T^*E^5$ :

$$ds_5^2 = (1 + h(x)/4)^2 dr^2 + r^2 d\omega_4^2, \tag{4.3}$$

where the metric is to be evaluated at  $r = r_0$ . In this rather round-about way, we have lifted the conformal fluctuations of the metric on  $T^*S^4$  to fluctuations of the metric on the co-tangent bundle  $T^*E^5$ .

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<sup>10</sup>This is not the most convenient choice for understanding renormalizability, but we are at present only concerned with the issue of stability.

To extend this to include the fluctuations  $h_{\alpha\beta}^\perp$  is straightforward since these are tensors built on the co-tangent bundle  $T^*S^4$ , which is represented in  $E^5$  by co-vectors normal to  $e^r(x^\mu)$  for all  $x^\mu$ . Thus, fluctuations confined to  $T^*S^4$  are unchanged in passing to  $T^*E^5$ . Altogether then, in the TT-gauge, eq. (4.1), the fluctuations of the metric in  $T^*S^4$  may be represented by fluctuations of the metric of  $T^*E^5$  implicitly defined in spherical coordinates by

$$ds_5^2 = (1 + h(x)/4)^2 dr^2 + \left(g_{\alpha\beta}^B + h_{\alpha\beta}^\perp\right) dx^\alpha dx^\beta, \tag{4.4}$$

with the understanding that  $\alpha, \beta$  refer to coordinates on  $TS^4$  or  $T^*S^4$ . On-shell, we must set  $r = r_0$ .

As can easily be seen in Cartesian coordinates, the symmetries of  $E^5$  are the Poincaré semigroup  $SO(5) \times P^5$ , where  $P^5$  represents translations. The generators of translations,  $P^5$ , commute with each other but transform as a vector  $\mathbf{5}$  under  $SO(5)$ . These isometries may also be inferred from the Killing equations, eq. (2.6), which, in  $E^5$ , become

$$\nabla_i \xi_j + \nabla_j \xi_i = 0. \tag{4.5}$$

In Cartesian coordinates, where the spin connection vanishes, the general solution is  $\xi^i = \omega^i_j x^j + k^i$ , for constants  $k^i$  and antisymmetric matrix  $\omega^{ij}$ . The first term represents the 10 generators of  $SO(5)$  rotations; the second term, of 5 translations. Since  $x^i \omega_{ij} x^j = 0$ , the rotation generators  $\omega^i_j x^j$  are orthogonal to the radial vector  $x^i$ ; they lie within the tangent bundle  $TS^4$ .

In going on-shell,  $r = r_0$ , the translation symmetry is broken but  $SO(5)$  is preserved. We shall return to the consequences of the breaking in the next section, but a major benefit of the lifting of fluctuations from  $S^4$  to  $E^5$  is that the decomposition of  $h(x)$  in terms of irreducible representations (irreps) of  $SO(5)$  is far simpler in flat five-dimensional  $E^5$  than on four-dimensional  $S^4$ . As discussed in appendix B, the basis functions for these irreps correspond to harmonics  $f_n(x)$  on  $E^5$  that depend on the components of the single vector  $x^i \partial_i$ . In Cartesian coordinates, each harmonic is a polynomial of degree  $n$  satisfying Laplace's equation  $\square_5 f_n = \sum_i \partial^2 f_n / \partial x^{i2} = 0$ . The irreps  $f_n(x)$  are given by the symmetric, traceless tensors  $\mathcal{S}^{ijk\dots}(x)$  of degree  $n$ . In spherical coordinates on  $E^5$ , they take the form  $f_n^m = r^n \phi_n^m(\omega^\alpha)$ , where we added an index  $m$  that runs over the number of linearly independent tensors of degree  $n$ . The  $\phi_n^m(\omega^\alpha)$  are spherical harmonics on the unit  $S^4$ , which, conventionally, are taken to be an orthonormal basis of functions. Any nonsingular scalar field on  $S^4$ , such as  $h(x)$ , may be expanded as  $h(x) = \sum h_n^m \phi_n^m(\omega^\alpha)$ .

Some further details of these representations are given in appendix B and have been reviewed in ref. [15]. Suffice to say that the five zero modes of particular interest to us correspond to  $n = 1$ , having  $\phi_1^m(\omega^\alpha) = x^m / r_0$ ,  $\{m = 1, \dots, 5\}$ . The  $\phi_1^m$  can be taken to be proportional to the five functions  $x^m / r$  in eq. (A.1). The question is why these particular modes  $\sum h_1^m \phi_1^m(\omega^\alpha)$  turn out to be zero modes.

## 5 The five zero mode fluctuations

In the embedding  $S^4 \rightarrow E^5$ , we assigned  $S^4$  to the submanifold  $\delta_{ij}x^i x^j = r_0^2$ , but we could equally well have chosen the submanifold

$$\delta_{ij}(x^i - b^i)(x^j - b^j) = r_0^2, \tag{5.1}$$

for some constant five-vector  $b^i$ . From the point of view of  $E^5$ , this corresponds to the  $S^4$  sphere centered at  $b^i$  rather than at the origin; call it  $S_{\mathbf{b}}^4$ . The classical action will be the same on  $S_{\mathbf{b}}^4$  as on  $S_0^4$  because the two manifolds are diffeomorphic. This can be explicitly seen, for example, by replacing  $x^i$  by  $x^i - b^i$  in the definition of spherical coordinates, eq. (A.1). The metric on  $E^5$ , eq. (3.1), is invariant under translations, so the projection onto  $S_{\mathbf{b}}^4$  will give the same functional form  $g_{\mu\nu}(\omega^\alpha)$  in spherical coordinates as its projection onto the original sphere  $S_0^4$ . Thus,  $b^i$  are moduli characterizing the embedding in  $E^5$  but having no physical relevance to the dynamics on  $S_{\mathbf{b}}^4$ .

For infinitesimal  $\Delta b^i$ , the first-order change in the submanifold  $S_0^4$  is  $x^2 - 2\Delta b_i x^i = r_0^2$ , so locally, the change appears to be a radial displacement or, better, a rescaling<sup>11</sup> of the radius by  $(1 + \Delta b^r/r_0)$ . Clearly, a change of embedding from  $b^i$  to  $b^i + \Delta b^i$  in eq. (5.1) is identical to a change  $x^i \rightarrow x^i - \Delta b^i$  at every point on  $S_{\mathbf{b}}^4$ . So, instead of regarding this as a change of  $b^i$ , we may instead think of it as a change of  $x^i$  or, in the QFT, for fixed  $x^i$ , a fluctuation in the metric of the form  $(1 + h/4)^2 dr^2$ . It is equivalent to an infinitesimal change<sup>12</sup>  $\Delta h = -4\Delta b_i \hat{x}^i/r_0$ . This is precisely the form of the SO(5) vector fluctuation  $\Delta h_5 = \Delta h_i \hat{x}^i$ , with  $\Delta h_i = -4\Delta b_i/r_0$ .

This last statement, which agrees with the intuitive idea that an infinitesimal translation in some direction involves both radial and tangential displacements, is worth expanding upon. Even though locally, the rescaling is radial, it must of course be true for every point on  $S^4$ . Consequently, if  $v_r(\omega^\mu) = 0$  for all points on  $S^4$ , then<sup>13</sup>  $v_i = 0$  on  $S^4$  for all  $i$ . Since  $v_r \equiv v_i e_r^i$  and  $e_\alpha^i = r_0 \partial_\alpha e_r^i$ , eq. (A.4), it follows that  $\partial_\alpha v_r = (\partial_\alpha v_i) e_r^i + (v_i/r_0) e_\alpha^i = 0$ . Since  $e_r^i$  and  $e_\alpha^i$  are orthogonal, each term must vanish separately. Hence,  $v_\alpha = 0$  for all angles  $\omega^\alpha$ , so  $\mathbf{v} = 0$  on  $S^4$  in any frame.

Therefore, this particular coherent fluctuation is equivalent to an infinitesimal displacement of the entire manifold, under which, according to the preceding analysis, the action is invariant. This is *why* these five vector modes in  $h_5$  are zero modes. In terms of the metric on  $S^4$ , eq. (4.1), we know that these correspond to conformal fluctuations. Indeed, we show in appendix C that these infinitesimal translations also correspond to conformal Killing fields, which are not usually associated with symmetries of the action. We also show that the conformal Killing fields can be associated locally with SO(5) rotations, so they in fact do generate zero modes.

<sup>11</sup>This scaling may be seen embedding  $S^4$  into the light-cone in six-dimensional Minkowski space  $M^{5,1}$ , for which the isometries are the conformal group, SO(5,1). See, e.g., appx. D of ref. [24].

<sup>12</sup>In the notation used earlier and in appendix B,  $\hat{x}^m = \phi_1^m(\omega^\alpha)$ , the five spherical harmonics.

<sup>13</sup>This argument breaks down where our spherical coordinates become singular, but one can patch these to another set of spherical coordinates with the axes rotated.

If we regard the  $S_{\mathbf{b}}^4$  sphere as having a fixed center  $\mathbf{b}$  when embedded in flat five-dimensional space, then these zero modes would not be allowed fluctuations of the metric. This strongly suggests that these coherent fluctuations associated with  $h_{\mathfrak{S}}$  are unphysical.

The preceding arguments do not depend upon the quadratic approximation to the fluctuations, and, since it is based on the underlying dS symmetries, it is plausible that the argument would extend to all orders in perturbation theory for the effective action, eq. (2.1). These zero modes are a consequence of the maximal global symmetry assumed for the background.

Of course, if this background were unstable, then the assumption of global  $SO(5)$  symmetry would become questionable. At the least, it is necessary that, at a local extremum, the quadratic approximation should have no negative eigenvalues. It is well-known that the fluctuations in E-H gravity do have negative eigenvalues, so whether or not there are also zero modes is somewhat of a moot point, but these same zero modes do occur there. For renormalizable gravity, Avramidi [3, 4] observed that, for a range of the couplings  $a, b$  there were in fact no unstable modes and that the only zero modes were the five discussed here. In contrast to his conclusion, however, we do not expect these zero modes to be removed in higher order, so it remains to be explained how they should be handled.

If we do not integrate out the fluctuations, the second-order calculation constitutes a test of whether the Euclidean action is classically stable. To the extent that we may regard rigid translations in five-dimensions of the dS submanifold as unphysical, these coherent zero modes do not in fact constitute a flat direction in the physically allowed space of fluctuations. In that case, we may conclude that the Euclidean classical EOM are in fact stable for a range of the couplings.

Returning to the QFT, one integrates over the second-order fluctuations to obtain the one-loop corrections to the classical action. To make use of such a calculation requires going off-shell in order to be able to take variational derivatives to determine corrections to the EOM and correlation functions. Such calculations are inherently gauge-dependent, although the value of the action on-shell is not. Further, the EPI off-shell cannot be analytically performed except in cases when the background field is assumed to have a high degree of symmetry. Assuming the background field retains maximal symmetry, only the scalar curvature  $R$  needs to be determined. (This calculation is analogous to calculating the effective potential in ordinary, flat-space field theory.) It has been carried out [3, 4] for a range of gauges. Off-shell, there appear to be no zero modes, at least for some gauge choices, and the one-loop correction can be carried out in a neighborhood of the classical curvature. The vanishing of the first variation then determines the one-loop correction to  $R$ .

Christensen and Duff [1] calculated the value of the one-loop corrections to the effective action in E-H gravity assuming an *arbitrary* background field but using the classical EOM,  $R_{\mu\nu} = \Lambda g_{\mu\nu}$ , in order to obtain a gauge-invariant result. (A generic background field presumably has no zero modes.) They also performed the calculation in a maximally symmetric background and showed that agreement with their first calculation, when restricted to dS background, is obtained only if the zero modes, including the five non-isometric conformal modes, are properly accounted for. We do not doubt this conclusion; these zero modes are certainly present, even for the E-H effective field theory, but their interpretation

is a matter for further discussion. As we have argued, they are a legacy of a symmetry between distinct but identical spacetimes in  $E^5$ .

Unlike more familiar applications [25], including the one involving the partition function for Schwarzschild black holes treated in section 3 of ref. [15], the collective coordinates  $b^i$  associated with these coherent fluctuations, analogous to their  $q^m$ , are not dynamical coordinates associated with metric fluctuations and do not reflect a physically realizable flat direction.

## 6 Some comments on Lorentzian signature

Although we shall leave the case of Lorentzian signature for future work, we shall indicate some of the differences and challenges that occur and offer some conjectures about what we expect to find. There is no problem starting from the classical solution without choosing coordinates. Embedded in five-dimensional Minkowski space, the dS solution corresponds to the hyperbolic submanifold  $H^4$  described by  $x^i x^j \eta_{ij} = r_0^2$ , where  $\eta_{ij} = \text{Diag}\{1, 1, 1, 1, -1\}$ . The Minkowski metric  $ds^2 = \eta_{ij} dx^i dx^j$ , has global isometries associated with the Poincaré semigroup,  $\text{SO}(4, 1) \rtimes P^5$ , which is broken to  $\text{SO}(4, 1)$  on the hyperboloid  $H^4$ . The hyperboloid has topology  $R \times S^{n-1}$ ; the main difference from the Euclidean case is that the manifold is no longer compact.

As before, we can discuss the diffeomorphic family of hyperboloids  $H_{\mathbf{b}}^4$  defined by  $(x^i - b^i)(x^j - b^j)\eta_{ij} = r_0^2$ , for fixed vector  $b^i$ . There will no doubt be zero modes of the fluctuations of each manifold associated with infinitesimal changes of  $b^i$ . We would expect that their treatment should be analogous to the Euclidean case; they represent unphysical fluctuations.

There are two things one would like to investigate for the Lorentzian case, viz., unitarity and the possible role of Euclidean instantons [26, 27]. Concerning unitarity, the definition of the Hilbert space of states and the associated norm is frame dependent already in a curved background, even before quantizing gravity. (See, e.g., refs. [28, 29].) The definition of the Hilbert space of states is usually associated with fixed time slices (spacelike submanifolds with timelike normal,) but there are a great many possibilities because the global symmetry is maximal. The natural choice for discussing unitarity in a Hamiltonian framework would be static coordinates:

$$ds^2 = - \left(1 - \frac{\rho^2}{r_0^2}\right) dt^2 + \left(1 - \frac{\rho^2}{r_0^2}\right)^{-1} d\rho^2 + \rho^2 d\omega_2^2, \tag{6.1}$$

for  $0 \leq \rho < r_0$ . These coordinates obviously develop singularities at the cosmological horizon  $\rho = r_0$  and therefore only cover a portion of the dS manifold. This situation is very much like the horizon for the Schwarzschild black hole in static coordinates, which can only be reached asymptotically, but this is a property of the frame and not a singularity of the manifold. Unlike the BH, one can find other frames that do cover the entire manifold without encountering a true spacetime singularity. For example, in global coordinates,

$$ds^2 = -dt^2 + r_0^2 \cosh^2(t/r_0) d\omega_3^2, \tag{6.2}$$

in which the isometry  $R \times S^3$  is manifest. The existence of global coordinates that cover the entire manifold is another difference from the Euclidean case. Normalizability at fixed time  $t$  once again reduces to normalization on a compact manifold  $S^3$ . How to demonstrate unitarity in such time-dependent backgrounds is unclear. Further, the usual definition of a well-defined no-particle state (“vacuum”) includes cluster decomposition of correlation functions, something that appears to be impossible on a compact manifold.

On the other hand, in a semi-classical approximation, it may be possible to understand the effects of the instanton as a tunneling amplitude between  $SO(4)$  coverings of  $S^3$  in the distant past with coverings in the distant future. If so, the Gauss-Bonnet coupling constant may acquire a dynamical significance analogous to the  $\Theta$  parameter in QCD. Since  $SO(4) \cong SU(2) \otimes SU(2)$ , the parallels may be very close, as the topology of  $SU(2)$  is<sup>14</sup>  $S^3$ . We hope to return to these questions in the future.

## 7 Discussion

Zero modes of scalar fields in curved spacetime have been discussed for a long time. (See, e.g., refs. [28, 29].) One lesson learned is that free massless scalars can sometimes be misleading and ought to be examined as limits of interacting QFTs or in the context of dynamical gravity. Minimal coupling (absence of a  $\phi^2 R$  interaction), for example, is not a fixed point of any nonsupersymmetric, interacting theory, so such models [19] should be examined with care [30].

Folacci [20] has considered a simplified version of the problem of the five zero modes in dS background.<sup>15</sup> He associates the zero modes with a “five-dimensional gauge transformation”  $h(x) \rightarrow h(x) + h_{\mathfrak{5}}$ . The effective action depends only upon the background field,  $g_{\alpha\beta}^B$  not on  $h(x)$ , but it is certainly true that, to one loop order, if  $h_{\mathfrak{5}}$  is expressed in terms of the five spherical harmonics on  $S^4$ , the on-shell effective action does have the symmetry  $\Gamma[g_{\alpha\beta}^B] = \Gamma[g_{\alpha\beta}^B(1 + h_{\mathfrak{5}}(r_0, \vartheta_\alpha))^2]$ . (See eq. (4.4).) A gauge-symmetry is designed to remove an unphysical degree of freedom in local field theory, but this local symmetry is a remnant of the global symmetries and would not be present in another background. As discussed in the Introduction, section 1, these modes are peculiar to spacetimes with maximal global symmetry. Because these are non-isometric zero modes, we cannot even associate a conserved current with them. It is hard to see how to associate this property with a gauge symmetry of the action on  $S^4$ .

Folacci [20] also suggests that the zero modes are a consequence of the compactness of the sphere  $S^4$  of Euclidean dS and would not be present for the non-compact hyperboloid  $H^4$  of Lorentzian dS. Consequently, he argues, they present a barrier to analytic continuation from Euclidean to Lorentzian signature. It is true that on a compact manifold, modes may be allowable whose Lorentzian analogs would be non-normalizable because the spacetime volume becomes infinite. However, as indicated in the preceding section, the normalization of states should be performed on fixed time-slices, for which the metric is

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<sup>14</sup>The two  $SU(2)$ 's may be called left and right, and there may well be chiral representations related to the Hirzebruch signature, and instantons associated with this index [26] as well.

<sup>15</sup>See also the appendix to ref. [19].

compact (except for the so-called Poincaré slice). Further, we can explicitly turn spherical coordinates on  $S^4$ , eq. (A.1), into global coordinates on  $H^4$ , eq. (6.2), by replacing  $\vartheta_1 \rightarrow \pi/2 - it$ . There is no difficulty normalizing the conformal Killing modes at fixed  $t$ .

Follacci [20] further argues that the S-matrix will be infrared-divergent. Even if there were an S-matrix, the same may be said about quantum electrodynamics (QED) in Minkowski space, for which the true asymptotic states of the theory are orthogonal to the Fock states. This “IR catastrophe” is not insurmountable [31]. The asymptotic states non-relativistically are Coulomb wave functions and, relativistically, are probably coherent states [32, 33]. In any case, these IR divergences do not prevent QED from making contact with the real world. The infinity of Fock states must be summed up to form observables having a finite energy resolution. The IR divergences in gravity are no worse than in QED [34] (at least not in asymptotically flat spacetimes). So, even though Follacci’s arguments may be formally correct, it must be shown that the IR divergences prevent predictions analogous to those of QED, when rephrased in terms of the limits of hypothetical measurements of limited accuracy. Since there is no S-matrix when  $\Lambda \neq 0$ , we must study long-time, long-distance correlation functions on-shell. Further work is needed to determine just what the infrared sensitivity of dS spacetime implies and how Lorentzian correlation functions may be related to their Euclidean counterparts.

Although it is often said that there are no local observables in gravity, in reality, all measurements are determined by the apparatus used. Theory may be used to relate them to the distant past (e.g., in astrophysics and cosmology) but both observationally and theoretically, calculations involve gauge-invariant correlations over finite times and distances. The actual measurements single out special “frames” and, from the point of view of the path integral, select a particular set of histories.<sup>16</sup> Such correlation functions will not be IR divergent, but, to complete the story, one must investigate the character of the dependence on large distances and long times in order to establish that observables in dS spacetime can be expressed in terms of the accuracy of the measuring apparatus.

## 8 Conclusions

Clearly, there is more that must be done to clarify these infrared issues for Lorentzian signature, but these matters seem to be essentially unrelated to the zero modes of interest here. More generally, the nature of measurement introduces apparatus that selects nearly classical histories of one sort or another, so that spacetime events decohere. (See footnote 16.) This will necessarily break exact dS invariance, so this discussion may be delicate but hopefully will be controllable in a manner similar to QED.

It has sometimes been suggested<sup>17</sup> that these modes are a feature at one-loop and unlikely to be sustained in higher-order. Since we have a renormalizable theory of gravity, we ought to be able to answer this, at least in principle. Nobody has done calculations beyond one-loop order, but our arguments in section 5 depend only upon the symmetries of the background and not on the order in the loop expansion. Unless some unsta-

<sup>16</sup>See, e.g., Gell-Mann & Hartle [35] and references therein.

<sup>17</sup>See, e.g., ref. [4] and the summary of earlier literature in ref. [20].

ble modes arise in higher-order, our conclusions should be good to all orders. Since the renormalizable theory is asymptotically free, in fact, the one-loop approximation ought to be good at high scales, so the absence of negative modes should not be undermined by higher-order corrections.

Analogous phenomena may occur in other models in which the background (or condensate) is assumed to have certain continuous global isometries. Simply subtracting zero modes should not be done without understanding their origin.

The assumption of exact dS symmetry is not correct in any relevant cosmological application since the presence of matter will lead to a stress-energy tensor that will contribute a background energy density and pressure for which the equation of state differs from that associated with a cosmological constant. Thus, more realistic cosmologies (such as the  $\Lambda$ -Cold Dark Matter model or, more generally, a Friedmann-Lemaître-Robertson-Walker metric) will break dS symmetry, and these non-isometric zero modes will disappear since they depend crucially upon the assumption that the background topology is  $S^4$  or  $H^4$ .

There are other reasons to doubt that dS is the correct background in any realistic cosmology. The dS metric assumes that the isometries are eternal, and it is unlikely that its symmetry between past and future is correct for applications to our universe. For example, inflationary cosmologies suggest that the dS approximation is only good for a finite period of time, having both an initial time when the exponential expansion begins and a final time when it effectively ends. For example, assuming that there is a finite initial time in the distant past only after which the dS metric becomes a good approximation is already a significant modification, which is sufficient to cure some infrared problems [36, 37]. Whether there is any sense to the other times described by the dS metric, for example the period of contraction rather than expansion, depends on speculations about the universe before the big bang, which may or may not have observable consequences for our universe. Most cosmologists, at least those exploring inflation, assume that a better approximation to dS spacetime is, during inflationary expansion, to adopt Poincaré coordinates and take only half of the dS manifold. That seems plausible although it would be nice not to identify the approximation with a special coordinate frame. With the addition of matter, the questions become more complicated since other fields may condense. However, we have seen that these zero modes remain present, not only in the models considered by others, but also in all the models that we have examined [13, 21, 22]. There is every reason to expect that they remain to the extent that the background is well approximated by dS space.

For renormalizable gravity, having argued that these zero modes are unphysical and do not represent flat directions, we can conclude that dS space is perturbatively stable for some range of couplings. This will remain correct in the presence of matter, at least so long as all couplings are asymptotically free [13]. In a separate publication [38], we shall discuss these matters further and explore the spectrum and, to a limited extent, the meaning of unitarity.

Asymptotically-free, renormalizable gravity can, at worst, be used to suggest some new cosmological possibilities, or, at best, to provide a consistent extension of quantum gravity within QFT. Having shown that there are models that are asymptotically free in all couplings that do not require fine-tuning [13], it may even be that renormalizable

gravity is a consistent completion of Einstein gravity. Demonstrating unitarity remains the outstanding problem.

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## A Basic concepts and notation

Euclidean  $dS^4$  can be depicted as the four-sphere  $S^4$  of radius  $r_0 = \sqrt{12/R_0}$ , where  $R_0$  is the on-shell value of the scalar curvature. The isometries of  $dS^4$  are most easily displayed by embedding  $S^4$  in  $E^5$ . On  $E^5$ , we imagine setting up five Cartesian coordinate axes. Tangents to the coordinate axes form vectors denoted by  $\partial_i \equiv \partial/\partial x^i$ , which form the basis of a vector space. Each point  $x$  in  $E^5$  can be assigned coordinates  $x^i$  in  $R^5$  according to the decomposition<sup>18</sup>  $x = x^i \partial_i$ . The duals to  $\partial_i$  are denoted by  $dx^i$ , linear functionals or one-forms on  $E^5$ , with  $dx^i[\partial_j] \equiv \delta_j^i$ . The five  $dx^i$  form a basis for the vector space  $T^*E^5$  of one-forms,  $da = a_i dx^i$ . (One-forms are frequently referred to as covariants, in contrast to vectors, which are sometimes called contravariants.)

$E^5$  is equipped with the metric  $\delta_{ij}$ , the components of the symmetric, covariant tensor  $ds^2 = \delta_{ij} dx^i dx^j$ , where  $dx^i dx^j$  stands for the direct product  $dx^i \otimes dx^j$ . This may be used to define the standard norm  $\|v\| = \sqrt{\delta_{ij} v^i v^j}$ . This implies the usual Cartesian inner product of vectors  $\vec{v} \cdot \vec{w} \equiv \delta_{ij} v^i w^j$ . The direct product is to be contrasted with the antisymmetric exterior product or two-form  $\alpha^{ij} = dx^i \wedge dx^j = -dx^j \wedge dx^i$ .

The symmetries of  $E^5$  are the Poincare group  $SO(5) \times P^5$ , a 15-dimensional group consisting of arbitrary translations of a point together with rotations in five dimensions. Of course, one may go on to discuss other coordinate systems on  $E^5$ , such as cylindrical, parabolic, elliptic, bipolar, etc. For example, spherical coordinates may be defined on  $E^5$  by

$$\begin{aligned} x^1 &= r \cos \vartheta_1, \\ x^2 &= r \sin \vartheta_1 \cos \vartheta_2, \\ x^3 &= r \sin \vartheta_1 \sin \vartheta_2 \cos \vartheta_3, \\ x^4 &= r \sin \vartheta_1 \sin \vartheta_2 \sin \vartheta_3 \cos \vartheta_4, \\ x^5 &= r \sin \vartheta_1 \sin \vartheta_2 \sin \vartheta_3 \sin \vartheta_4, \end{aligned} \tag{A.1}$$

where  $0 < \vartheta_\alpha < \pi$   $\{\alpha = 1, 2, 3\}$ ,  $0 \leq \vartheta_4 < 2\pi$ . The corresponding metric on  $E^5$  then takes the form

$$ds^2 = dr^2 + r^2 d\omega_4^2, \tag{A.2a}$$

$$d\omega_4^2 = d\vartheta_1^2 + \sin^2 \vartheta_1 d\vartheta_2^2 + \sin^2 \vartheta_1 \sin^2 \vartheta_2 d\vartheta_3^2 + \sin^2 \vartheta_1 \sin^2 \vartheta_2 \sin^2 \vartheta_3 d\vartheta_4^2. \tag{A.2b}$$

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<sup>18</sup>Concerning nomenclature,  $E^5$  without the metric is often referred to as  $R^5$ , although the notation in the literature is not uniform and sometimes these are used interchangeably.

The singular character of coordinates is revealed by the vanishing of  $g \equiv \det g_{\alpha\beta}$ . For these spherical coordinates, the determinant is

$$g_5^{dS} = r^8 \sin^6 \vartheta_1 \sin^4 \vartheta_2 \sin^2 \vartheta_3, \quad (\text{A.3})$$

which is obviously singular at  $r = 0$  or when any of these three  $\vartheta_i = 0, \pi$ . This restricts this coordinate patch on  $E^5$  to exclude these values.

An orthonormal basis in five dimensions in spherical coordinates can be chosen as

$$e_r^i = x^i/r = (c_1, s_1 c_2, s_1 s_2 c_3, s_1 s_2 s_3 c_4, s_1 s_2 s_3 s_4), \text{ and } e_\alpha^i, \text{ where} \quad (\text{A.4a})$$

$$e_1^i = r \partial_{\vartheta_1} e_r^i = r(-s_1, c_1 c_2, c_1 s_2 c_3, c_1 s_2 s_3 c_4, c_1 s_2 s_3 s_4), \quad (\text{A.4b})$$

$$e_2^i = r \partial_{\vartheta_2} e_r^i = r s_1(0, -s_2, c_2 c_3, s_2 c_3 c_4, s_2 s_3 s_4), \quad (\text{A.4c})$$

$$e_3^i = r \partial_{\vartheta_3} e_r^i = r s_1 s_2(0, 0, -s_3, c_3 c_4, c_3 s_4), \quad (\text{A.4d})$$

$$e_4^i = r \partial_{\vartheta_4} e_r^i = r s_1 s_2 s_3(0, 0, 0, -s_4, c_4), \quad (\text{A.4e})$$

where we have abbreviated  $c_k \equiv \cos \vartheta_k, s_k \equiv \sin \vartheta_k, k = 1, \dots, 4$ . These fünfbein satisfy  $e_\mu^i e_\nu^j \delta_{ij} = g_{\mu\nu}$ . The inverse of the matrix  $[e_\mu^i]$  will be written as  $[e_i^\mu]$ , so it is necessary to adhere to our notational conventions using Latin indices for Cartesian coordinates and Greek indices for spherical coordinates. (An exception is the use of  $r$  rather than  $\rho$  for the radial coordinate.)

As mentioned in the text, one can replace  $x^i$  by  $x^i - b^i$  in eq. (A.1) for any constant five-vector  $\mathbf{b}$ , without making any changes in the metric on a subdomain at fixed  $r_0 = \|\mathbf{x} - \mathbf{b}\|$ ; consequently, the value of the classical action, e.g., eq. (2.2), on  $S_{\mathbf{b}}^4$  is independent of  $\mathbf{b}$ .

As a brief refresher on the Cartan formalism, an exterior differential  $d$  takes a  $p$ -form  $\alpha$  to a  $p+1$ -form denoted  $d\alpha$ . Recall that the exterior differential of a function  $f(x)$  (0-form) is the usual differential  $df[x] = dx^i \partial_i f$ , discussed above. If  $\alpha = a_i dx^i$  is a one-form, then  $d\alpha \equiv da_i \wedge dx^i = (\partial a_i / \partial x^j) dx^j \wedge dx^i$  in any coordinates. Similarly, for an arbitrary  $p$ -form. The exterior derivative  $d$  has the property that  $d^2\alpha = 0$  on any form  $\alpha$ .

The interior product or contraction operator is an operation  $\iota_v$  associated with a vector  $v$  that takes a  $p$ -form  $\alpha$  into a  $p-1$ -form according to  $\iota_v[\alpha] \equiv \alpha[v, \dots]$ , which symbolically means “evaluate the  $p$ -form on the vector  $v$ .” On a zero-form (i.e, a function)  $f(x)$ ,  $\iota_v[f] \equiv 0$ . On a one-form, such as  $df$ ,  $\iota_v[df] \equiv df[v] = v^i \partial_i f$ , the usual directional derivative. For a two-form, e.g.,  $\alpha = df \wedge dg$ ,  $\iota_v[\alpha] \equiv df[v] \wedge dg - df \wedge dg[v] = v^i (\partial_i f dg - df \partial_i g)$ , etc. This generalizes in an obvious way to arbitrary  $p$ -forms. Like the exterior differential, this is a coordinate-independent operation having the property that  $\iota_v^2 \alpha = 0$  on any form  $\alpha$ .

## B The SO(5) spherical harmonics

We very briefly review the SO(5) spherical harmonics  $(n, 0)$ , which are all that are needed in this paper. (For further discussion, see ref. [15] and references therein.) In Cartesian coordinates, these functions are formed from the five-vector  $x^i$ :  $f_n(x) \equiv f_{i_1 i_2 \dots i_n} x^{i_1 i_2 \dots i_n}$ , with  $f_{i_1 i_2 \dots i_n}$  a constant, symmetric, traceless co-tensor.  $f_0$  is just a constant.  $f_1$  takes the form  $f_1 = f_k x^k$ .  $f_2 = f_{ij} x^i x^j$ , with  $f_i^i = 0$ , etc. These can be associated with the irreducible representations (irreps) frequently labeled by their dimensions:

$\{\mathbf{1}, \mathbf{5}, \mathbf{14}, \dots\}$ . These irreps are harmonics, i.e., solutions of Laplace’s equation in five dimensions,  $\square_5 f_n = 0$ . (For Lorentzian signature, the d’Alembertian replaces the Laplacian.)

In spherical coordinates, these take the form  $f_n(r, \omega^\alpha) = r^n \phi_n(\omega^\alpha)$ , where  $\omega^\alpha$  denotes the four angles implicitly defined in eq. (A.1). Using the metric from eq. (A.2), we may write Laplace’s equation as

$$-\square_5 f_n = \left[ -\frac{1}{r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} L^2 \right] r^n \phi_n(\omega^\alpha) = 0, \tag{B.1}$$

where  $L^2$  denotes the quadratic Casimir<sup>19</sup> of “orbital” angular momentum in five-dimensions. Explicitly,  $L^2 \equiv \sum_{i<j} L_{ij}^2$ , with  $L_{ij} dx^i \wedge dx^j$  the 10 generators of SO(5). In Cartesian coordinates,  $L_{ij} = -i(x_i \partial_j - x_j \partial_i)$ . Carrying out the radial derivatives and evaluating on the  $S^4$  submanifold  $r = r_0$  yields

$$-\square_{S^4} \phi_n^m(\omega^\alpha) = \frac{1}{r_0^2} L^2 \phi_n^m(\omega^\alpha) = \frac{n(n+3)}{r_0^2} \phi_n^m(\omega^\alpha), \tag{B.2}$$

where  $r_0^2 = 12/R_0$ ,  $R_0$  being the curvature of  $S^4$ , and  $m$  labels the linearly independent functions having a common eigenvalue. Thus, the “spherical harmonics”  $\phi_n^m(\omega^\alpha)$  obey

$$L^2 \phi_n^m(\omega^\alpha) = n(n+3) \phi_n^m(\omega^\alpha). \tag{B.3}$$

These symmetric irreps are sometimes called the  $(n, 0)$  representations (because a second integer  $(n, p)$  is needed to delineate all representations). It is a combinatoric exercise [39] to determine that the degree of degeneracy of eigenvalue  $n(n+3)$  is

$$d_n = \frac{1}{6} (n+1)(n+2)(2n+3), \tag{B.4}$$

the dimension of the representation. As a check, the non-isometric zero modes correspond to  $n = 1$ , for which  $d_1 = 5$ .

On  $S^4$ , with  $r = r_0$ , the Cartesian coordinates are not intrinsically well-defined, but we may continue making reference to the ambient space by using the angular variables  $\vartheta_\alpha$  to label points on  $S^4$ . In other words, since  $x^i x^j \delta_{ij} = r_0^2$ , only four of the five coordinates  $x^i$  are independent on  $S^4$ . Similarly, we may continue using the vierbein  $e_\alpha$ , eq. (A.4), as a local basis of the tangent space  $TS^4_x$ . Just as we denoted  $e_r$  by  $e_r^i = \hat{x}^i$ , it is convenient to continue using five-component notation for  $e_\alpha^i$  in order to avoid having to specify the choice of coordinates on  $S^4$ . Further, the  $e_r^i = \phi_1^i(\omega^k)$  do transform five- vectors under SO(5) rotations.

We have pointed out in eq. (3.4) that the Hodge dual of  $e_r$  in  $E^5$  is proportional to the four-form associated with the volume  $dV_4$  on  $T^*S^4$ . Since  $dV_4$  is coordinate invariant, it is a gauge invariant, but the identification with the conformal rescaling of the metric in the unitary TT-gauge is gauge dependent. For example, in the unimodular gauge,  $\sqrt{g_4} = 1$ , the association would be quite different. The conformal Killing equation however is gauge covariant.

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<sup>19</sup> $L^2$  is sometimes called the spherical Laplacian or the Laplace-Beltrami operator on the sphere.

## C Killing vectors and Killing forms

In this section, we elaborate on the disposition of the five Killing vectors on  $E^5$  resulting from translation invariance when restricted to an  $S^4$  submanifold. We know that there are five zero modes of the fluctuations on  $S^4$ , but we wish to understand how they might be related, if at all, to the isometries of  $E^5$ , the Poincaré semigroup. Translation invariance is manifest in Cartesian coordinates, and the corresponding Killing equations are

$$\nabla_i \xi_j + \nabla_j \xi_i = 0. \quad (\text{C.1})$$

Since the metric is simply  $\delta_{ij}$ , the spin connection vanishes, so that  $\nabla_i \xi_j = \partial_i \xi_j$ . This implies that the five components  $\xi_i = k_i$  for arbitrary constants  $k_i$  generate translations. Alternatively, since the inverse metric is  $\delta^{ij}$ , we may say that  $k^i \partial_i$  is a Killing vector for an arbitrary constants  $k^i$ . However, we are interested in  $S^4$ , which is not translation invariant, and it is not at all clear whether any of these are projected onto Killing vectors on the  $S^4$  submanifold.

To facilitate the connection between  $E^5$  and  $S^4$ , let us rewrite the Killing equations, eq. (C.1), in spherical coordinates, eq. (A.1),

$$\nabla_r \xi_r = 0, \quad (\text{C.2a})$$

$$\nabla_r \xi_\beta + \nabla_\beta \xi_r = 0, \quad (\text{C.2b})$$

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0. \quad (\text{C.2c})$$

(Recall that, with the exception of the radius  $r$ , we use Latin indices for Cartesian coordinates, and Greek indices for spherical coordinates on  $S^4$ . In eq. (C.2), we have abbreviated the angular components  $\omega^\alpha$ , with a slight abuse of notation, simply by the index  $\alpha$ .) Although the submanifold of special interest has radius  $r = r_0$ , for the time being, we can take any fixed value of  $r$ . In spherical coordinates, the connection is non-trivial. Noting the metric eq. (A.2), the nonzero connections in spherical coordinates take the form

$$\Gamma_{\alpha\beta}{}^r = -\frac{g_{\alpha\beta}}{r}, \quad \Gamma_{r\beta}{}^\alpha = \frac{\delta_\beta^\alpha}{r}, \quad \Gamma_{\alpha\beta}{}^\lambda, \quad (\text{C.3})$$

where  $\Gamma_{\alpha\beta}{}^\lambda$  are the connections on the  $S^4$  submanifold at fixed  $r$ . Although their precise form will not be needed, we note that  $\Gamma_{\alpha\beta}{}^\lambda$  is independent of  $r$ .

The first equation above, eq. (C.2a), becomes  $\partial_r \xi_r = 0$ , so that  $\xi_r = \xi_r(\omega^\alpha)$ , independent of  $r$ . This agrees with our orthonormal basis in  $E^5$ , eq. (A.4), since  $k_r = k_i e_r^i(\omega^\alpha)$  is independent of  $r$ . Using this in the second equation, eq. (C.2b), we see that  $\xi_\beta$  must be linear in  $r$ , in agreement with  $\xi_\beta = k_i e_\beta^i$ . Thus,  $\nabla_r \xi_\beta = \partial_r \xi_\beta - \xi_\beta/r = 0$ . Therefore, this equation implies that each covariant derivative vanishes separately,  $\nabla_\beta \xi_r = \partial_\beta \xi_r - \xi_\beta/r = 0$ , which agrees with eq. (A.4). Finally, eq. (C.2c) is not quite the same as the corresponding equations on  $S^4$ , because the connection on  $T^*E^5$  differs from the connection on  $T^*S^4$ ,<sup>20</sup>

$$\nabla_\alpha \xi_\beta = (\nabla_\alpha \xi_\beta)_4 - \Gamma_{\alpha\beta}{}^r \xi_r = (\nabla_\alpha \xi_\beta)_4 + \frac{g_{\alpha\beta}}{r}, \quad (\text{C.4})$$

<sup>20</sup>Eq. (C.4) is a special case of Gauss's equation in which the second term on the right-hand side is associated with the second fundamental form on  $S^4$  embedded in  $E^5$ .

where, by definition,  $(\nabla_\alpha \xi_\beta)_4$  only involves the  $S^4$  connection  $\Gamma_{\alpha\beta}^\lambda$ . Thus, eq. (C.2c) becomes

$$(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha)_4 = -2 \frac{g_{\alpha\beta}}{r}. \quad (\text{C.5})$$

This implies  $(\nabla_\alpha \xi^\alpha)_4 = -4/r$ , so that the preceding equation may also be expressed as

$$(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha)_4 = (\nabla \cdot \xi)_4 \frac{g_{\alpha\beta}}{2}. \quad (\text{C.6})$$

We may set  $r = r_0$  here<sup>21</sup> to conclude that the Killing forms for infinitesimal translations  $\xi_\alpha$  on  $T^*E^5$  project onto non-isometric, conformal Killing forms on  $T^*S^4$ .

The components  $\xi_\alpha$  of a co-vector in spherical coordinates are related to the corresponding co-vector  $k_i$  in Cartesian coordinates according to  $\xi_\alpha = k_i e_\alpha^i$ . Since  $k_i$  are arbitrary constants, this implies that there are five conformal Killing forms  $e_{\vartheta_\beta}^m d\vartheta_\beta$  on  $T^*S^4$ , where the index  $m$  may be identified with the Cartesian components  $e_\beta^m$ .

The preceding concerns the properties of the dS background and not the fluctuations directly, but the generators of isometries have consequences for the infinitesimal fluctuations and implications for zero modes. We have seen that none of the translation generators  $\xi^\alpha$  in the tangent bundle  $TS^4$  are true Killing vectors, so they do not generate isometries. (The interpretation of  $\xi_r$  for fluctuations on  $S^4$  is discussed in section 3.) Therefore, none of these observations imply that any fluctuation is directly associated with a zero mode of the action on  $S^4$ , but we have presented such an argument in section 5.

That argument did not require this result on the conformal Killing forms although it is a corollary. Even though these  $e_{\vartheta_j}^m d\vartheta_j$  do correspond to zero modes, they are not really new. Recall that the most general Killing form on  $E^5$  is  $\omega_{ij} x^j + k^i$ , with the antisymmetric constants  $\omega_{ij}$  corresponding to the 10 rotation generators of  $\text{SO}(5)$ . For a fixed direction  $x^j$ , there appear to be five nontrivial rotation generators  $\omega_{ij} x^j$ , but, since the radial projection onto  $e_r^i$  vanishes, there are only four non-trivial rotations<sup>22</sup> at fixed  $x^i$  on  $S^4$ . We may denote them by  $\omega_{r\vartheta_k} = \omega_{ij} e_r^i e_{\vartheta_k}^j = -\omega_{\vartheta_k r}$ . At fixed  $r$ , the rotation group  $\text{SO}(5)$  remains a good symmetry, so these four rotations do reflect true Killing forms at fixed  $x^i$ . and, therefore, do correspond to zero modes. Even though the translations  $\xi_i$  are not isometries on a fixed  $r$  submanifold, their projection onto the four-sphere of radius  $r$  can be compensated by a rotation, viz., one may choose  $\omega_{r\vartheta_k}$  such that

$$(\xi_j + \omega_{ij} r e_r^i) e_{\vartheta_k}^j = \xi_{\vartheta_k} + r \omega_{r\vartheta_k} = 0, \quad (\text{C.7})$$

for each  $\vartheta_k$ . Paradoxically, the conformal Killing forms  $\xi_\alpha$  may locally be written as a sum of true Killing forms. Since the linear combination depends upon the direction  $\hat{x}$ , this is understandable. The really surprising result is encoded in the five ‘‘radial’’ zero modes which, we argued, reflect displacement of the center of the  $S^4$  sphere.

<sup>21</sup>These relations hold for *any* fixed value of  $r$ , an observation that proves useful in classically scale-invariant models [13, 21, 22], wherein the value of the scalar curvature is classically undetermined.

<sup>22</sup>The little group of a fixed point on  $S^4$  is  $\text{SO}(4)$ , the 6 generators that annihilate the normal  $\hat{x}^i = e_r^i$  at that point. This leaves  $10 - 6 = 4$  non-trivial rotations isomorphic to the cosets  $\text{SO}(5)/\text{SO}(4)$ .

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